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UPGRADED METHODS FOR THE EFFECTIVE COMPUTATION OF MARKED SCHEMES ON A STRONGLY STABLE IDEAL

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ABSTRACT. Let $J \subset S = K[x_0, \dots, x_n]$ be a monomial strongly stable ideal. The collection $\mathcal{M}\mathrm{f}(J)$ of the homogeneous polynomial ideals I, such that the monomials outside J form a K-vector basis of S/I, is called a *J-marked family*. It can be endowed with a structure of affine scheme, called a *J-marked scheme*. For special ideals J, J-marked schemes provide an open cover of the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, where $p(t)$ is the Hilbert polynomial of S/J. Those ideals more suitable to this aim are the *m-truncation* ideals $\underline{J}_{\geq m}$ generated by the monomials of degree $\geq m$ in a saturated strongly stable monomial ideal $\underline{J}$. Exploiting a characterization of the ideals in $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ in terms of a Buchberger-like criterion, we compute the equations defining the $\underline{J}_{\geq m}$-marked scheme by a new reduction relation, called *superminimal reduction*, and obtain an embedding of $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ in an affine space of low dimension. In this setting, explicit computations are achievable in many non-trivial cases. Moreover, for every m, we give a closed embedding $\phi_m : \mathcal{M}\mathrm{f}(\underline{J}_{\geq m}) \hookrightarrow \mathcal{M}\mathrm{f}(\underline{J}_{\geq m+1})$, characterize those ϕ_m that are isomorphisms in terms of the monomial basis of $\underline{J}$, especially we characterize the minimum integer m_0 such that ϕ_m is an isomorphism for every $m \geq m_0$.

INTRODUCTION

Let J be a monomial ideal of the polynomial ring $S = K[x_0, \dots, x_n]$ in $n+1$ variables over a field K. In this paper, we refine and develop the study begun in [7] to characterize the homogeneous polynomial ideals $I \subset S$ such that the monomials outside J form a K-vector basis of the K-vector space S/I. If J is strongly stable, such homogeneous ideals constitute a family $\mathcal{M}\mathrm{f}(J)$, that is called a J-marked family and that can be endowed in a very natural way with a structure of affine scheme, called a J-marked scheme, which turns out to be homogeneous with respect to a non-standard grading and flat at J (see [7]). Moreover, J-marked schemes generalize the notion of Gröbner strata [15] because $\mathcal{M}\mathrm{f}(J)$ contains all the ideals having J as initial ideal with respect to some term order; however in general $\mathcal{M}\mathrm{f}(J)$ contains also ideals which do not belong to a Gröbner stratum.

In this paper we focus on a particular class of strongly stable ideals: letting $\underline{J}$ be a saturated strongly stable ideal, we will consider the truncations $\underline{J}_{\geq m}$, for every positive integer m, because in this setting marked schemes give a theoretical and effective alternative to the study of Hilbert schemes as subvarieties of a Grassmannian. Theorem 3.3 and Example 3.4 show the reason for the choice of this special setting. Let $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ be the Hilbert scheme that parameterizes all subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$, r be the Gotzmann number of $p(t)$ and $q(t) = |S_t| - p(t) = \binom{n+t}{n} - p(t)$ be the volume polynomial. By theoretical results in [4, 7, 6] we are able to compute first the set $\mathcal{B}_{p(t)}$ of all saturated strongly stable ideals $\underline{J}$ in S, such that $p(t)$ is the Hilbert polynomial of $S/\underline{J}$; then, for every ideal $\underline{J} \in \mathcal{B}_{p(t)}$, we compute explicit equations of degree $\leq deg(p(t)) + 2$ defining $\mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$ as an affine subscheme of $\mathbb{A}^{p(r)q(r)}$. In particular, every $\mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$ can be embedded in $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ as an open subscheme and moreover, as $\underline{J}$ varies in $\mathcal{B}_{p(t)}$, the $\underline{J}_{\geq r}$-marked schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$ form an open cover of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, up to changes of coordinates in $\mathbb{P}^n$. Observe that this is not true for $\mathcal{M}\mathrm{f}(\underline{J})$, because in

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general $\mathcal{M}\mathrm{f}(\underline{J})$ is not isomorphic to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ (see Example 6.1 and [24, Section 5]). Such computational method is effective because the dimension $p(r)q(r)$ of the affine space in which the $\underline{J}_{\geq r}$-marked schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$ are embedded is significantly lower than the number $\binom{|S_r|}{q(r)}$ of Plücker coordinates. However there is room for further significant improvements.

The present paper is inspired by two questions raised, on the one hand, by similarities of marked schemes with Gröbner strata and, on the other hand, by experimental observations on examples.

First, we observed that we could eliminate a significant number of variables from the equations defining $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ as an affine subscheme of $\mathbb{A}^{p(m)q(m)}$, computed using the method developed in [7]; in this way we obtain equations of higher degree than the starting ones, but often more convenient to use (for example, see [7, Appendix]). This feature has already been observed and studied for Gröbner strata in [15]. The bottleneck is that elimination of variables is too time-consuming. From this we wondered how to obtain this new set of equations using in the computations only necessary variables, avoiding the elimination process.

Our second observation is that, for a fixed $\underline{J} \in \mathcal{B}_{p(t)}$, as the integer m grows, the families parameterized by marked schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ become larger, up to a certain value of m bounded by r. The study of relations among marked schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ as m varies can improve the efficiency of the computational methods in [7]: indeed, if $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ and $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m'})$ are isomorphic for some integers $m' < m$, then we can choose to compute defining equations that involve a lower number of variables, that is equations for $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m'}) \subseteq \mathbb{A}^{p(m')q(m')}$. In particular, for applications to the study of Hilbert schemes, we would like to determine a priori the minimum integer m_0 for which $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m_0})$ is isomorphic to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, that is $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m_0}) \simeq \mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$.

In this paper, considering truncated ideals $\underline{J}_{\geq m}$, we answer to both questions by a new reduction algorithm, called *superminimal reduction*, that uses, for every $I \in \mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$, its $\underline{J}_{\geq m}$-*superminimal basis* (see Definition 3.9), a special subset of the $\underline{J}_{\geq m}$-marked basis of I.

For every strongly stable monomial ideal J, the notion of J-marked basis (Definition 1.8) is the main tool for the study of marked schemes in [7] and also the starting point of the present paper. Indeed a homogeneous ideal I belongs to $\mathcal{M}\mathrm{f}(J)$ if and only if I is generated by a J-marked basis G (Proposition 1.11). This basis resembles a reduced Gröbner basis for I, where J plays the role of the initial ideal and the strongly stable property plays the role of the term order.

Indeed, similarly to a reduced Gröbner basis, G is a system of generators of I that contains a polynomial f_α for every term x^α in the monomial basis of J: $f_\alpha = x^\alpha - T(f_\alpha)$ where no monomial appearing in $T(f_\alpha)$ belongs to J. Moreover, G is characterized by a Buchberger-like criterion (Theorem 2.11) and allows to compute the J-reduced form modulo I of every polynomial in S, by a Noetherian reduction process (Proposition 2.3).

The J-superminimal basis of I introduced in the present paper is a special subset sG of G containing a polynomial for every term in the monomial basis of the saturated ideal $\underline{J}$ (for the details, see Definitions 3.5 and 3.9): the two sets G and sG are equal if and only if $\underline{J} = J$. Using only polynomials in sG and the strongly stable property of J, we define a special process of reduction $\xrightarrow{sG*}$ called superminimal reduction.

In the special case when J is a truncation $\underline{J}_{\geq m}$ of a saturated strongly stable ideal $\underline{J}$, the $\underline{J}_{\geq m}$-superminimal basis sG has very interesting properties. First of all in this case (but not in general) the superminimal reduction $\xrightarrow{sG*}$ turns out to be Noetherian (see Theorem 3.14, (i) and Example 3.13). Moreover, although in general sG is not a system of generators of I, it completely determines the ideal I because we can solve the ideal-membership problem by the superminimal reduction process $\xrightarrow{sG*}$ (Theorem 3.14, (iv)). This allows to compute equations for $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ as a subscheme of an affine space of dimension far lower than $p(m)q(m)$, without any variable elimination process (Theorem 5.4), answering the first question above.

In this new setting, in Theorem 5.7 we compare the $\underline{J}_{\geq m}$-marked schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ for a fixed saturated $\underline{J}$ as m varies, using superminimal bases. We prove that for every m there is a closed scheme-theoretical embedding $\phi_m : \mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1}) \hookrightarrow \mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$. Moreover, we provide an easy criterion on the monomial basis of $\underline{J}$ to characterize the integers m for which ϕ_m is an isomorphism. Especially, this criterion allows to determine the minimum integer m_0 such that ϕ_m is an isomorphism for every $m \geq m_0$, and in particular $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m_0})$ is isomorphic to an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ (see [4]).

Our investigation on marked schemes lies in the framework of the methods and results obtained in the last years by several authors [5, 10, 15, 20, 23, 24] about families of ideals with a fixed monomial basis for the quotient. Another close framework is the one in [2, 19], where the authors study the collection of all monomial ideals J that are initial ideals of a fixed homogeneous ideal I w.r.t. some term order.

In [4], the results of [7] and the ones of the present paper are applied to study relations among marked schemes and Hilbert schemes; in particular, in [4] the authors study how marked schemes can be used to obtain a computable open cover of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ that has also interesting theoretical features. We are confident that these results, both theoretical and computational ones, may be helpful in the solution of some open problems about Hilbert schemes; indeed, they have been already applied in order to investigate the locus of points of the Hilbert scheme with bounded regularity (see [1]); the ideas and strategies used by [11] to study deformations of ACM curves are inspired by the ones in the present paper; in [16] the authors apply the computational strategy to the Hilbert scheme of locally CM curves. Other investigations led by these tools are in progress. In the future, we are interested in deeply comparing marked bases with other kinds of Gröbner-like bases, referring to [18].

In Section 1, we introduce notations and basic results and, in Section 2, we recall the Buchber-ger-like criterion described in [7], with some development that involves the Eliahou and Kervaire syzygies of a strongly stable ideal (Theorem 2.11, (iii) and Corollary 2.13). Moreover, we compute sets of generators of the ideal $\mathfrak{A}_J$ that defines the structure of affine scheme of $\mathcal{M}\mathrm{f}(J)$ (see Corollary 2.17 and Remark 2.18).

In Section 3, we define the superminimal reduction (Definition 3.11) and investigate its properties. In Section 4 we describe a new Buchberger-like criterion for $\underline{J}_{\geq m}$-marked bases (Theorem 4.5) and some variants of it (Corollary 4.6 and Theorem 4.7). In particular, the second variant leads to a remarkable improvement of the efficiency of explicit computational procedures.

In Section 5, we focus on the ideal that defines the structure of affine scheme of $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ and we characterize the integers m, m', $m > m'$, such that the schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ and $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m'})$ are isomorphic (Theorem 5.7).

Finally, in Section 6 we provide examples in which we apply the proved results and we compute the equations defining the affine structure of a $\underline{J}_{\geq m}$-marked scheme in a "small" affine space, using the Algorithm that we describe in the Appendix.

1. Notations and generalities

Let K be an algebraically closed field and $S := K[x_0, \dots, x_n]$ ($K[x]$ for short) the polynomial ring in $n+1$ variables with $x_0 < \cdots < x_n$. We will denote by $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ every monomial in S, where $\alpha = (\alpha_0, \dots, \alpha_n)$ is its multi-index and $|\alpha|$ is its degree.

We say that a monomial x^γ is divisible by x^α ($x^\alpha \mid x^\gamma$ for short) if there exists a monomial x^β such that $x^\alpha \cdot x^\beta = x^\gamma$. If such monomial does not exist, we will write $x^\alpha \nmid x^\gamma$. For every monomial $x^\alpha \neq 1$, we set $\min(x^\alpha) := \min\{x_i : x_i \mid x^\alpha\}$ and $\max(x^\alpha) := \max\{x_i : x_i \mid x^\alpha\}$.

We will denote by $>_{\mathtt{Lex}}$ the usual lexicographic order on the monomials of S: in our setting $x^\alpha >_{\mathtt{Lex}} x^\beta$ if the last non-null element of $\alpha - \beta$ is positive.

We consider the standard grading on $S = \bigoplus_{m\in\mathbb{Z}} S_m$, where S_m is the additive group of homogeneous polynomials of degree m; we let $S_{\geqslant m} = \bigoplus_{m'\geq m} S_{m'}$ and in the same way, for every subset $A \subseteq S$, we let $A_m = A \cap S_m$ and $A_{\geqslant m} = A \cap S_{\geqslant m}$. Elements and ideals in S are always supposed to be homogeneous.

We will say that a monomial x^β can be obtained by a monomial x^α through an *elementary move* if $x^\alpha x_j = x^\beta x_i$ for some variables $x_i \neq x_j$. In particular, if $i < j$, we say that x^β can be obtained by x^α through an *increasing* elementary move and we write $x^\beta = \mathrm{e}^+_{i,j}(x^\alpha)$, whereas if $i > j$ the move is said to be *decreasing* and we write $x^\beta = \mathrm{e}^-_{i,j}(x^\alpha)$. The transitive closure of the relation $x^\beta > x^\alpha$ if $x^\beta = \mathrm{e}^+_{i,j}(x^\alpha)$ gives a partial order on the set of monomials of a fixed degree, that we will denote by $>_B$ and that is often called *Borel partial order*:

$$x^\beta >_B x^\alpha \iff \exists\, x^{\gamma_1}, \dots, x^{\gamma_t} \text{ such that } x^{\gamma_1} = \mathrm{e}^+_{i_0,j_0}(x^\alpha),\ \dots\ , x^\beta = \mathrm{e}^+_{i_t,j_t}(x^{\gamma_t})$$

for suitable indexes i_k, j_k. In analogous way, we can define the same relation using decreasing moves:

$$x^\beta >_B x^\alpha \iff \exists\, x^{\delta_1}, \dots, x^{\delta_s} \text{ such that } x^{\delta_1} = \mathrm{e}^-_{h_0,l_0}(x^\beta),\ \dots\ , x^\alpha = \mathrm{e}^-_{h_s,l_s}(x^{\delta_s})$$

for suitable indexes i_k, j_k. Note that every term order $\succ$ is a refinement of the Borel partial order $>_B$, that is $x^\beta >_B x^\alpha$ implies that $x^\beta \succ x^\alpha$.

Definition 1.1. An ideal $J \subset K[x]$ is said to be *strongly stable* if every monomial x^β such that $x^\beta >_B x^\alpha$, with $x^\alpha \in J$, belongs to J.

A strongly stable ideal is always *Borel fixed*, that is fixed by the action of the Borel subgroup of upper triangular matrices of $GL(n+1)$. If $ch(K) = 0$, also the vice versa holds (e.g. [8]) and [13] guarantees that in generic coordinates the initial ideal of an ideal I, w.r.t. a fixed term order, is a constant Borel fixed monomial ideal called the *generic initial ideal* of I.

If J is a monomial ideal in S, B_J will denote its monomial basis and $\mathcal{N}(J)$ its *sous-escalier*, that is the set of monomials not belonging to J.

An homogeneous ideal I is *m-regular* if the i-th syzygy module of I is generated in degree $\leq m+i$, for all $i \geq 0$. The *regularity* of I is the smallest integer m for which I is m-regular; we denote it by $\mathrm{reg}(I)$. The *saturation* of a homogeneous ideal I is $I^{\mathrm{sat}} = \{f \in S \mid\ \forall\, j = 0, \dots, n, \exists\, r \in \mathbb{N} : x_j^r f \in I\}$. The ideal I is *saturated* if $I^{\mathrm{sat}} = I$ and is *m-saturated* if $(I^{\mathrm{sat}})_t = I_t$ for each $t \geq m$. The *satiety* of I is the smallest integer m for which I is m-saturated; we denote it by $\mathrm{sat}(I)$.

We recall that if J is strongly stable then $\mathrm{reg}(J) = \max\{\deg x^\alpha \ :\ x^\alpha \in B_J\}$ [3, Proposition 2.9] and $\mathrm{sat}(J) = \max\{\deg x^\alpha \ :\ x^\alpha \in B_J \text{ and } x_0 \mid x^\alpha\}$ (for example, see [14, Corollary 2.10]).

Lemma 1.2. *Let J be a strongly stable ideal in $K[x_0, \dots, x_n]$. Then:*

(i) $x^\alpha \in J \setminus B_J \ \Rightarrow\ \dfrac{x^\alpha}{\min(x^\alpha)} \in J$;

(ii) $x^\beta \in \mathcal{N}(J)$ *and* $x_i x^\beta \in J \ \Rightarrow$ *either* $x_i x^\beta \in B_J$ *or* $x_i > \min(x^\beta)$.

Proof. Both properties follow from Definition 1.1. □

Definition 1.3. For every monomial x^α in S we denote by $x^{\underline{\alpha}}$ the monomial obtained putting $x_0 = 1$. Analogously, if J is a monomial ideal in $K[x]$, we denote by $\underline{J}$ the ideal in $K[x]$ generated by $\{x^{\underline{\alpha}} \ :\ x^\alpha \in B_J\}$.

If J is strongly stable, then $J^{\mathrm{sat}} = \underline{J}$ (this follows straightforwardly from [14, Corollary 2.10]); in particular, the set $\{x^{\underline{\alpha}} \ :\ x^\alpha \in B_J\}$ of the monomials $x^{\underline{\alpha}}$, such that $x^\alpha = x^{\underline{\alpha}} \cdot x_0^t$ belongs to B_J for a suitable $t \geq 0$, contains the monomial basis $B_{\underline{J}}$.

Many tools we are going to use were introduced in [22] and developed in [7]. For this reason, we now resume some notations and definitions given in those papers.

Definition 1.4. For any non-zero homogeneous polynomial $f \in S$, the *support* of f is the set $\mathrm{Supp}(f)$ of monomials that appear in f with a non-zero coefficient.

Definition 1.5 ([22])**.** A *marked polynomial* is a polynomial $f \in S$ together with a specified monomial of $\mathrm{Supp}(f)$ that will be called *head term* of f and denoted by $\mathrm{Ht}(f)$.

Remark 1.6. Although in this paper we use the word "monomial", we say "head term" for coherency with the notation introduced by [22]. Anyway, in this paper there will be no possible ambiguity on the meaning of "head term of f", because we will always consider marked polynomials f such that the coefficient of $\mathrm{Ht}(f)$ in f is 1.

Definition 1.7 ([7])**.** Given a monomial ideal J and an ideal I, a polynomial is *J-reduced* if its support is contained in $\mathcal{N}(J)$ and a *J-reduced form modulo I* of a polynomial h is a polynomial h_0 such that $h - h_0 \in I$ and $\mathrm{Supp}(h_0) \subseteq \mathcal{N}(J)$. If there is a unique J-reduced form modulo I of h, we call it *J-normal form modulo I* and denote it by $\mathrm{Nf}(h)$.

Note that every polynomial h has a unique J-reduced form modulo an ideal I if and only if $\mathcal{N}(J)$ is a K-basis for the quotient S/I or, equivalently, $S = I \oplus \langle \mathcal{N}(J)\rangle$ as a K-vector space. If moreover I is homogeneous, the J-reduced form modulo I of a homogeneous polynomial is supposed to be homogeneous too. These facts motivate the following definitions.

Definition 1.8. A finite set G of homogeneous marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $\mathrm{Ht}(f_\alpha) = x^\alpha$, is called a *$J$-marked set* if the head terms $\mathrm{Ht}(f_\alpha)$ form the monomial basis B_J of a monomial ideal J, are pairwise different and every x^γ belongs to $\mathcal{N}(J)$, i.e. $|\mathrm{Supp}(f_\alpha) \cap J| = 1$. We call *tail* of f_α the polynomial $T(f_\alpha) := \mathrm{Ht}(f_\alpha) - f_\alpha$, so that $\mathrm{Supp}(T(f_\alpha)) \subseteq \mathcal{N}(J)$. A J-marked set G is a *J-marked basis* if $\mathcal{N}(J)$ is a basis of $S/(G)$ as a K-vector space.

Definition 1.9. The collection of all the homogeneous ideals I such that $\mathcal{N}(J)$ is a basis of the quotient S/I as a K-vector space will be denoted by $\mathcal{M}\mathrm{f}(J)$ and called a *J-marked family.* If J is a strongly stable ideal, then $\mathcal{M}\mathrm{f}(J)$ can be endowed with a natural structure of scheme (see [7, Section 4]) that we call *J-marked scheme*.

Remark 1.10.

(i) The ideal (G) generated by a J-marked basis G has the same Hilbert function of J, hence $\dim_K J_m = \dim_K (G)_m$, by the definition of J-marked basis itself. Moreover, note that a J-marked basis is unique for the ideal that it generates, by the uniqueness of the J-normal forms modulo I of the monomials in B_J.
(ii) $\mathcal{M}\mathrm{f}(J)$ contains every homogeneous ideal having J as initial ideal w.r.t. some term order, but it might also contain other ideals: see [7, Example 3.18].
(iii) When J is a strongly stable ideal, all homogeneous polynomials have J-reduced forms modulo every ideal generated by a J-marked set G (see [7, Theorem 2.2]).

Proposition 1.11. *Let J be a strongly stable ideal, I be a homogeneous ideal generated by a J-marked set G. The following facts are equivalent:*

(i) $I \in \mathcal{M}\mathrm{f}(J)$
(ii) G is a J-marked basis;
(iii) $\dim_k I_t = \dim_K J_t$*, for every integer t;*
(iv) if $h \in I$ and h is J-reduced, then $h = 0$.

Proof. For the equivalence among the first three statements, see [7, Corollaries 2.3, 2.4, 2.5]. For the equivalence among (i) and (iv), observe that if $I \in \mathcal{M}\mathrm{f}(J)$, then every polynomial has a unique J-reduced form modulo I; so, the J-reduced form modulo I of a polynomial of I must be null. Vice versa, it is enough to show that every polynomial f has a unique J-reduced form modulo I. Let $\bar{f}$ and $\bar{\bar{f}}$ be two J-reduced forms modulo I of f. Then, $\bar{f} - \bar{\bar{f}}$ is a J-reduced polynomial of I because $f - \bar{f}$ and $f - \bar{\bar{f}}$ belong to I by definition. We are done, because $\bar{f} - \bar{\bar{f}}$ is null by the hypothesis. □

2. Background on Buchberger-like criterion for J-marked bases and some developments

In this section we recall and develop some results of [7]. Throughout this section, *J is a strongly stable ideal and G is a J-marked set.*

Definition 2.1. Let $m_J := \min\{t : J_t \neq (0)\}$ be the initial degree of J. For every $\ell \geq m_J$ we define the set

$$W_\ell := \{x^\delta f_\alpha \mid f_\alpha \in G \text{ and } \mid \delta + \alpha| = \ell\}$$

that becomes a set of marked polynomials by letting $\mathrm{Ht}(x^\delta f_\alpha) = x^{\delta+\alpha}$. We set $W = \cup_\ell W_\ell$. For every $\ell \geq m_J$ we also define a special subset of W_ℓ:

$$V_\ell := \{x^\delta f_\alpha \in W_\ell \mid x^\delta = 1 \text{ or } \max(x^\delta) \leq \min(x^\alpha)\}.$$

We let $V = \cup_\ell V_\ell$. Moreover, $\langle V \rangle$ denotes the vector space generated by the polynomials in V and $\xrightarrow{V_\ell}$ is the reduction relation on homogeneous polynomials of degree ℓ defined in the usual sense of Gröbner basis theory (see also [7, Proposition 3.6]).

The above Definition is equivalent to the Definition 3.2 in [7] due to Remark 3.3 of the same paper. Note that $(G)_\ell$ is generated by W_ℓ as a K-vector space.

Lemma 2.2. *Let J be a strongly stable ideal. An ideal I generated by a J-marked set G belongs to $\mathcal{M}\mathrm{f}(J)$ if and only if $\langle W \rangle = \langle V \rangle$ as K-vector spaces.*

Proof. It is sufficient to observe that for every $\ell \geq m_J$, the number of elements in V_ℓ is equal to the number of monomials in J_ℓ, so $\dim\langle V_\ell \rangle \leq \dim J_\ell$. On the other hand, $\dim\langle W_\ell \rangle = \dim I_\ell \geq \dim J_\ell$ by [7, Corollary 2.3]. By Proposition 1.11 we get the equivalence of the statements. □

We have already recalled that, when J is a strongly stable ideal, every homogeneous polynomial has a J-reduced form modulo an ideal generated by a J-marked set G (Remark 1.10 (iii)). Further, a J-reduced form of a homogeneous polynomial can be constructed by the reduction relation $\xrightarrow{V_\ell}$, as it is recalled by next Proposition.

Proposition 2.3. [7, Proposition 3.6] *With the above notation, every monomial $x^\beta \in J_\ell$ can be reduced to a J-reduced form modulo (G) in a finite number of reduction steps, using only polynomials of V_ℓ. Hence, the reduction relation $\xrightarrow{V_\ell}$ is Noetherian.*

The Noetherianity of the reduction relation $\xrightarrow{V_\ell}$ provides an algorithm that reduces every homogeneous polynomial of degree ℓ to a J-reduced form modulo (G) in a finite number of steps. We note that on the one hand it is convenient to substitute the polynomials in V_ℓ by their J-reduced normal forms for an efficient implementation of a reduction algorithm, but, on the other hand, in the proofs it is convenient to use the polynomials of V_ℓ as constructed in Definition 2.1.

2.1. **Order on W_ℓ.** Using the Noetherianity of the reduction relation $\xrightarrow{V_\ell}$, we can recognize when a J-marked set is a J-marked basis by a Buchberger-like criterion (see [7, Theorem 3.12]). To this aim we need to set an order on the set W_ℓ.

The order that we are going to define on W_ℓ in Definition 2.7 is based on the following Definition and Lemma that are inspired by [9] and [17, Lemma 2.11].

Definition 2.4. Given a strongly stable monomial ideal J in S, with monomial basis B_J, and a monomial $x^\gamma \in J$, we define

$$x^\gamma = x^\alpha *_J x^\eta, \qquad \text{with } \gamma = \alpha + \eta,\ x^\alpha \in B_J \text{ and } \min(x^\alpha) \geq \max(x^\eta).$$

This decomposition exists and is unique (see [9, Lemma 1.1]).

Lemma 2.5. *Let J be a strongly stable ideal. If x^ϵ belongs to $\mathcal{N}(J)$ and $x^\epsilon \cdot x^\delta = x^{\epsilon+\delta}$ belongs to J for some x^δ, then $x^{\epsilon+\delta} = x^\alpha *_J x^\eta$ with $x^\eta <_{\mathtt{Lex}} x^\delta$. Furthermore:*

(i) if $|\delta| = |\eta|$, then $x^\eta <_B x^\delta$; and
(ii) $x^{\underline{\eta}} <_{\mathtt{Lex}} x^{\underline{\delta}}$.

Proof. We can assume that x^δ and x^η are coprime; indeed, if this is not the case, we can divide the involved equalities of monomials by $\gcd(x^\delta, x^\eta)$. If $x^\eta = 1$, all the statements are obvious. If $x^\eta \neq 1$, then $\min(x^\delta) | x^\alpha$ because x^δ and x^η are coprime, hence $\min(x^\delta) \geq \min(x^\alpha) \geq \max(x^\eta)$ and so $\min(x^\delta) > \max(x^\eta)$ because they cannot coincide. This inequality implies both $x^\eta <_{\mathtt{Lex}} x^\delta$ and $x^{\underline{\eta}} <_{\mathtt{Lex}} x^{\underline{\delta}}$. Moreover, if $|\delta| = |\eta|$, this is also sufficient to conclude that $x^\eta <_B x^\delta$. $\square$

Remark 2.6. Observe that if $g_\beta = x^\delta f_\alpha$ belongs to V_ℓ, then $x^\beta = x^\alpha *_J x^\delta$.

Definition 2.7. Let $\geq$ be any order on G and $x^\delta f_\alpha$, $x^{\delta'} f_{\alpha'}$ be two elements of W_ℓ. We set

$$x^\delta f_\alpha \succeq_\ell x^{\delta'} f_{\alpha'} \Leftrightarrow x^\delta >_{\mathtt{Lex}} x^{\delta'} \text{ or } x^\delta = x^{\delta'} \text{ and } f_\alpha \geq f_{\alpha'}.$$

Lemma 2.8.

(i) For every two elements $x^\delta f_\alpha$, $x^{\delta'} f_{\alpha'}$ of W_ℓ we get

$$x^\delta f_\alpha \succeq_\ell x^{\delta'} f_{\alpha'} \Rightarrow \forall x^\eta : \quad x^{\delta+\eta} f_\alpha \succeq_{\ell'} x^{\delta'+\eta} f_{\alpha'},$$

where $\ell' = |\delta + \eta + \alpha|$.

(ii) Every polynomial $g_\beta \in V_\ell$ is the minimum w.r.t. $\preceq_\ell$ of the subset W_β of W_ℓ containing all polynomials of W_ℓ with x^β as head term.

(iii) If $x^\delta f_\alpha$ belongs to $W_\ell \setminus G_\ell$ and x^β belongs to $\mathrm{Supp}(x^\delta T(f_\alpha))$ with $g_\beta \in V_\ell$, then $x^\delta f_\alpha \succ_\ell g_\beta$.

Proof.

(i) This follows by the analogous property of the term order $>_{\mathtt{Lex}}$.

(ii) Let $g_\beta = x^{\delta'} f_{\alpha'}$ be the polynomial of V such that $x^\beta = x^{\alpha'} *_J x^{\delta'}$ and $x^\delta f_\alpha$ be another polynomial of W_β. We can assume that x^δ and $x^{\delta'}$ are coprime; otherwise, we can divide the involved inequalities of monomials by $\gcd(x^\delta, x^{\delta'})$. By Remark 2.6 and Definition 2.4, we have that $\max(x^{\delta'}) \leq \min(x^{\alpha'})$ and $\max(x^\delta) > \min(x^\alpha)$. Then, we get $\max(x^\delta) > \max(x^{\delta'})$ because $x^{\alpha'} \nmid x^\alpha$ and $x^\alpha \nmid x^{\alpha'}$. Thus, $x^\delta >_{\mathtt{Lex}} x^{\delta'}$.

(iii) If x^β belongs to B_J we are done. Otherwise, let $x^\beta = x^{\alpha'} *_J x^{\delta'}$ and note that every monomial of $\mathrm{Supp}(x^\delta f_\alpha)$ is a multiple of x^δ, in particular $x^\beta = x^{\delta+\gamma}$ for some $x^\gamma \in \mathcal{N}(J)$. By Lemma 2.5, we get $x^{\delta'} <_{\mathtt{Lex}} x^\delta$. $\square$

Remark 2.9. We point out that the order defined in [7, Definition 3.9] does not satisfy the conditions listed in Lemma 2.8 and in [7, Lemma 3.10]. These conditions have a crucial role in the proof of [7, Theorem 3.12] and for this reason it has been a mistake to use the order of [7, Definition 3.9] in that Theorem. So, here we replace such order by that defined in new Definition 2.7. Aside the order, the original reduction and Buchberger criterion are the same, as we will state in Theorem 2.11, (i) and (ii). Also, we give an improvement by Theorem 2.11, (iii) and by Corollary 2.13. Moreover, we observe that for the same reason the results about syzygies of the ideal I generated by a J-marked basis proposed in [7, Section 3] hold by using the order on W_ℓ of Definition 2.7 and do not hold by using the order of [7, Definition 3.9].

2.2. Improved Buchberger-like criterion for J-marked bases.

Definition 2.10. The *S-polynomial* of two elements f_α, $f_{\alpha'}$ of a J-marked set G is the polynomial $S(f_\alpha, f_{\alpha'}) := x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}$, where $x^{\gamma+\alpha} = x^{\gamma'+\alpha'} = lcm(x^\alpha, x^{\alpha'})$.

Theorem 2.11. (Buchberger-like criterion) *Let J be a strongly stable ideal and I the homogeneous ideal generated by a J-marked set G. With the above notation, TFAE:*

(i) $I \in \mathcal{M}\mathrm{f}(J)$;

(ii) $\forall f_\alpha, f_{\alpha'} \in G,\ S(f_\alpha, f_{\alpha'}) \xrightarrow{V_\ell} 0$;

(iii) $\forall f_\alpha, f_{\alpha'} \in G,\ S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'} = \sum a_j x^{\eta_j} f_{\alpha_j}$, *with* $x^{\eta_j} <_{\mathtt{Lex}} \max_{\mathtt{Lex}}\{x^\gamma, x^{\gamma'}\}$ *and* $x^{\eta_j} f_{\alpha_j} \in V_\ell$.

Proof. For the equivalence between (i) and (ii), we refer to the proof of [7, Theorem 3.12] by using Definition 2.7 instead of [7, Definition 3.9].

Statement (ii) implies (iii) by the definition of the reduction relation $\xrightarrow{V_\ell}$ and by Lemma 2.8 (iii). It remains to prove that statement (iii) implies (i).

We want to prove that $I = \langle V \rangle$ or, equivalently by Lemma 2.2, that $\langle V \rangle = \langle W \rangle$. It is sufficient to prove that $x^\eta \cdot V \subseteq \langle V \rangle$, for every monomial x^η. We proceed by induction on the monomials x^η, ordered according to $\mathtt{Lex}$. The thesis is obviously true for $x^\eta = 1$. We then assume that the thesis holds for any monomial $x^{\eta'}$ such that $x^{\eta'} <_{\mathtt{Lex}} x^\eta$.

If $|\eta| > 1$, we can consider any product $x^\eta = x^{\eta_1} \cdot x^{\eta_2}$, x^{η_1} and x^{η_2} non-constant. Since $x^{\eta_i} <_{\mathtt{Lex}} x^\eta, i = 1, 2$, we immediately obtain by induction

$$x^\eta \cdot V = x^{\eta_1} \cdot (x^{\eta_2} \cdot V) \subseteq x^{\eta_1} \langle V \rangle \subseteq \langle V \rangle.$$

If $|\eta| = 1$, then we need to prove that $x_i \cdot V \subseteq \langle V \rangle$. Since $x_0 V \subseteq V$, it is then sufficient to prove the thesis for $x^\eta = x_i$, assuming that the thesis holds for every $x^{\eta'} <_{\mathtt{Lex}} x_i$. We consider $g_\beta = x^\delta f_\alpha \in V$, where $\max(x^\delta) \le \min(x^\alpha)$. If $x_i g_\beta$ does not belong to V, then $\max(x_i \cdot x^\delta) > \min(x^\alpha)$, so $x_i > \min(x^\alpha)$. In particular, $x_i > \min(x^\alpha) \ge \max(x^\delta)$, so $x_i >_{\mathtt{Lex}} x^\delta$: by induction, it is now sufficient to prove the thesis for $x_i f_\alpha$.

We consider an S-polynomial $S(f_\alpha, f_{\alpha'}) = x_i f_\alpha - x^\gamma f_{\alpha'}$ such that $x^\gamma <_{\mathtt{Lex}} x_i$. Such S-polynomial always exists: for instance, we can consider $x_i x^\alpha = x^{\alpha'} *_J x^{\eta'}$. By the hypothesis $x_i f_\alpha - x^{\eta'} f_{\alpha'} = \sum a_j x^{\eta'_j} f_{\alpha_j}$ where $x^{\eta'} f_{\alpha'}, x^{\eta'_j} f_{\alpha_j} \in V_\ell$ and then $x_i f_\alpha$ belongs to $\langle V \rangle$. □

For any strongly stable ideal J, with monomial basis $B_J = \{x^{\alpha_1}, \ldots, x^{\alpha_r}\}$, we can consider the set of syzygies of the following kind

$$x_j e_{\alpha_i} - x^\eta e_{\alpha_k}, \quad \text{with } x_j > \min(x^{\alpha_i}) \ \text{ and } \ x_j x^{\alpha_i} = x^{\alpha_k} *_J x^\eta.$$

This set of syzygies is actually a minimal set of generators for the first module of syzygies of J; this is due to Eliahou and Kervaire (see [9] and [14, Theorem 1.31]).

Definition 2.12. We call *Eliahou-Kervaire couple* of the J-marked set G any couple of polynomials f_α, f_β, $\mathrm{Ht}(f_\alpha) = x^\alpha$, $\mathrm{Ht}(f_\beta) = x^\beta$, such that

$$x_j x^\alpha = x^\beta *_J x^\eta \text{ for some } x_j > \min(x^\alpha).$$

We call *Eliahou-Kervaire S-polynomial* (EK-polynomial, for short) of G an S-polynomial among an Eliahou-Kervaire couple of polynomials f_α and f_β. We denote such S-polynomial by $S^{EK}(f_\alpha, f_\beta)$. Observe that, thanks to the definition, an EK-polynomial is of kind

$$S^{EK}(f_\alpha, f_\beta) = x_j f_\alpha - x^\eta f_\beta, \text{ for some } x_j > \min(x^\alpha), \text{ with } x_j x^\alpha = x^\beta *_J x^\eta.$$

In the proof of Theorem 2.11, it is sufficient to assume that (iii) holds only for EK-polynomials, as stated in the following result.

Corollary 2.13. *With the same notation of Theorem 2.11,*

$$I \in \mathcal{M}\mathrm{f}(J) \Leftrightarrow \textit{ for every EK-polynomial between elements of } G,\ S^{EK}(f_\alpha, f_\beta) \xrightarrow{V_\ell} 0.$$

Proof. In the proof of Theorem 2.11 the crucial point is the existence of an S-polynomial of kind $x_i f_\alpha - x^\eta f_\beta$ with $x^\eta <_{\texttt{Lex}} x_i$, and we used exactly an EK-polynomial. □

2.3. **The scheme structure of $\mathcal{M}\mathrm{f}(J)$.** Now, we recall and develop some features of the affine scheme structure of $\mathcal{M}\mathrm{f}(J)$. Let $p(t)$ the Hilbert polynomial of S/J and r its Gotzmann number. In the following we will denote by $\mathcal{G}$ the J-marked set:

$$\mathcal{G} = \left\{ F_\alpha = x^\alpha - \sum C_{\alpha\gamma} x^\gamma : \mathrm{Ht}(F_\alpha) = x^\alpha \in B_J,\ x^\gamma \in \mathcal{N}(J)_{|\alpha|} \right\} \tag{1}$$

and by $\mathfrak{I}_J$ the ideal generated by $\mathcal{G}$ in the ring $K[C, x]$, where C is a compact notation for the set of new variables $C_{\alpha\gamma}$.

For every polynomial $H \in K[C, x]$, we denote by $\mathrm{Supp}_x(H)$ the set of monomials in the variables x_i that appear in H with non-null coefficients and by $\mathrm{Coeff}_x(H) \subset K[C]$ the set of such coefficients, that we call x-coefficients.

Let $\mathcal{V}_\ell$ and $\mathcal{W}_\ell$ be the analogous for $\mathcal{G}$ of V_ℓ and of W_ℓ, respectively, for any J-marked set G. We will denote by $\mathfrak{A}_J$ the ideal of $K[C]$ generated by the x-coefficients of the J-reduced forms, obtained by $\xrightarrow{\mathcal{V}_\ell}$, of the S-polynomials $S(F_\alpha, F_{\alpha'})$ among elements of $\mathcal{G}$. This ideal does not depend on $\xrightarrow{\mathcal{V}_\ell}$ and defines the subscheme structure of $\mathcal{M}\mathrm{f}(J)$ in the affine space $\mathbb{A}^{|C|}$ (see [7, Theorem 4.1]). Let $\mathfrak{A}_J^{EK}$ be the ideal of $K[C]$ generated by the x-coefficients of the J-reduced forms of the EK-polynomials of $\mathcal{G}$ obtained by $\xrightarrow{\mathcal{V}_\ell}$.

It is clear that $\mathfrak{A}_J^{EK} \subseteq \mathfrak{A}_J$. Anyway, we will prove that $\mathfrak{A}_J^{EK}$ and $\mathfrak{A}_J$ are the same ideal, although $\mathfrak{A}_J$ is defined by a set of generators bigger than the set of generators of $\mathfrak{A}_J^{EK}$. More precisely, we prove that the ideal $\mathfrak{A}_J^{EK}$ contains the x-coefficients of every J-reduced polynomial in $\mathfrak{I}_J$.

Lemma 2.14.

(i) For every monomial $x^\beta = x^\alpha *_J x^\delta \in J$, there is a formula of type

$$x^\beta = \sum a_i x^{\gamma_i} F_{\alpha_i} + H_\beta,$$

with $a_i \in K[C]$, $x^{\gamma_i} F_{\alpha_i} \in \mathcal{V}$, $x^{\gamma_i} \leq_{Lex} x^\beta$ and $\mathrm{Supp}_x(H_\beta) \subset \mathcal{N}(J)$.

(ii) For every polynomial $x_i F_\alpha \in \mathcal{W} \setminus \mathcal{V}$, there is a formula of type

$$x_i F_\alpha = \sum a_j x^{\eta_j} F_{\alpha_j} + H_{i,\alpha},$$

with $a_j \in K[C]$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}$, $x^{\eta_j} <_{\texttt{Lex}} x_i$, $\mathrm{Supp}_x(H_{i,\alpha}) \subset \mathcal{N}(J)$ and $\mathrm{Coeff}_x(H_{i,\alpha}) \subset \mathfrak{A}_J^{EK}$.

Proof. Statement (i) follows from the existence of J-reduced forms obtained by $\xrightarrow{\mathcal{V}_\ell}$ and by Lemma 2.8, (iii). Statement (ii) follows also from the definition of $\mathfrak{A}_J^{EK}$. □

Proposition 2.15. *For every polynomial $x^\delta F_\alpha \in \mathcal{W} \setminus \mathcal{V}$, we have*

$$x^\delta F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H_{\delta,\alpha}, \tag{2}$$

with $b_j \in K[C]$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}$, $x^{\eta_j} <_{\mathtt{Lex}} x^\delta$, $\mathrm{Supp}_x(H_{\delta,\alpha}) \subset \mathcal{N}(J)$ and $\mathrm{Coeff}_x(H_{\delta,\alpha}) \subset \mathfrak{A}_J^{EK}$.

Proof. For $|\delta| = 1$ it is enough to use Lemma 2.14 (ii). Assume that $|\delta| > 1$ and that the thesis holds for every $x^{\delta'} <_{\mathtt{Lex}} x^\delta$. Let $x_i = \min(x^\delta)$ and $x^{\delta'} = \frac{x^\delta}{x_i}$, so that $x^{\delta'} F_\alpha$ belongs to $\mathcal{W} \setminus \mathcal{V}$.

By the inductive hypothesis, we have $x^{\delta'} F_\alpha = \sum b'_j x^{\eta'_j} F_{\alpha_j} + H_{\delta',\alpha}$, with $x^{\eta'_j} <_{\mathtt{Lex}} x^{\delta'}$. So, multiplying by x_i, we obtain $x^\delta F_\alpha = \sum b'_j x_i x^{\eta'_j} F_{\alpha_j} + x_i H_{\delta',\alpha}$ and the thesis holds for every polynomial $x_i x^{\eta'_j} F_{\alpha_j}$ that belongs to $\mathcal{W} \setminus \mathcal{V}$ because $x_i x^{\eta'_j} <_{\mathtt{Lex}} x_i x^{\delta'} = x^\delta$. Then, we replace such polynomials by formulas of type (2) and obtain

$$x^\delta F_\alpha = \sum b_s x^{\eta_s} F_{\alpha_s} + H' + x_i H_{\delta',\alpha}$$

where the first sum satisfies the conditions of (2) and H' is J-reduced with $\mathrm{Supp}_x(H') \subset \mathcal{N}(J)$ and $\mathrm{Coeff}_x(H') \subset \mathfrak{A}_J^{EK}$.

Note that $\mathrm{Coeff}_x(x_i H_{\delta',\alpha}) = \mathrm{Coeff}_x(H_{\delta',\alpha}) \subset \mathfrak{A}_J^{EK}$, but we do not know if $\mathrm{Supp}_x(x_i H_{\delta',\alpha}) \subset \mathcal{N}(J)$. If $x^{\beta'} \in \mathrm{Supp}_x(H_{\delta',\alpha})$ has x-coefficient b in $H_{\delta',\alpha}$ and $x^\beta = x_i x^{\beta'}$ belongs to J, then we can use Lemma 2.14 (i) obtaining $bx^\beta = \sum b a_k x^{\gamma_i} F_{\alpha_k} + bH_\beta$. Moreover, if $x^\beta = x^{\alpha'} *_J x^\epsilon$, then $x^{\gamma_i} \leq_{\mathtt{Lex}} x^\epsilon <_{\mathtt{Lex}} x_i <_{\mathtt{Lex}} x^\delta$, where the second inequality is due to the fact that $x^{\beta'} \in \mathcal{N}(J)$ and to Lemma 2.5, and all the x-coefficients of H_β belong to $\mathfrak{A}_J^{EK}$ because they are divisible by b. Replacing all such monomials x^β, we obtain the thesis and $H_{\delta,\alpha}$ is J-reduced with x-coefficients in $\mathfrak{A}_J^{EK}$, because it is the sum of J-reduced polynomials with x-coefficients in $\mathfrak{A}_J^{EK}$. □

Corollary 2.16. *Every polynomial of $\mathfrak{I}_J$ can be written in a unique way as $\sum b_j x^{\eta_j} F_{\alpha_j} + H$, with $b_j \in K[C]$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}$ and H J-reduced. Moreover, we obtain also that $\mathrm{Coeff}_x(H) \subset \mathfrak{A}_J^{EK}$.*

Proof. By definition, every polynomial of $\mathfrak{I}_J$ is a linear combination of polynomials of $\mathcal{V} \cup (\mathcal{W} \setminus \mathcal{V})$ with x-coefficients in $K[C]$ and, by Proposition 2.15, every such polynomial can be written has described in the statement. Hence, we have only to prove the uniqueness of this writing.

Let $\sum b_j x^{\eta_j} F_{\alpha_j} + H = 0$ be the difference between two writings of the same polynomial of $\mathfrak{I}_J$, with $b_j \neq 0$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}$ pairwise different and H J-reduced. Let $x^{\eta_1} x^{\alpha_1}$ the maximum of the monomials w.r.t. the order for which $x^{\eta_i} x^{\alpha_i}$ is lower than $x^{\eta_j} x^{\alpha_j}$ if $x^{\eta_i} <_{\mathtt{Lex}} x^{\eta_j}$ or $x^{\eta_i} = x^{\eta_j}$ and $x^{\alpha_i} < x^{\alpha_j}$, where $<$ is any order fixed on B_J. By definition of $\mathcal{V}$, the unique polynomial of $\mathcal{V}$ with head term $x^{\eta_1} x^{\alpha_1}$ is $x^{\eta_1} F_{\alpha_1}$. Moreover, the monomial $x^{\eta_1} x^{\alpha_1}$ does not appear with a non-null coefficient in any polynomial of the sum because every other monomial belongs to $\mathcal{N}(J)$ or is lower than it, by construction. Further, $x^{\eta_1} x^{\alpha_1}$ does not belong to $\mathrm{Supp}_x(H)$ because $\mathrm{Supp}_x(H) \subset \mathcal{N}(J)$ and $x^{\eta_1} x^{\alpha_1} \in J$. Thus, we obtain a contradiction to the fact that $b_j \neq 0$. □

Corollary 2.17. *The ideal $\mathfrak{A}_J^{EK}$ contains the x-coefficients of every J-reduced polynomial of $\mathfrak{I}_J$. In particular, $\mathfrak{A}_J^{EK} = \mathfrak{A}_J$.*

Proof. Let F be a J-reduced polynomial of $\mathfrak{I}_J$ and let $F = \sum b_j x^{\eta_j} F_{\alpha_j} + H$ as in Corollary 2.16. Since F itself is J-reduced, also $F = 0 + F$ is a formula as described in Corollary 2.16 and we obtain that $F = H$, by the uniqueness of this formula. Hence, we have $\mathrm{Coeff}_x(F) = \mathrm{Coeff}_x(H) \subset \mathfrak{A}_J^{EK}$. The last assertion is due to the definition of $\mathfrak{A}_J$. □

Remark 2.18. Actually, for every ideal $\widehat{\mathfrak{A}}_J \subseteq \mathfrak{A}_J \subseteq K[C]$ such that condition (ii) of Lemma 2.14 holds, also Corollary 2.17 holds. We are then allowed to choose different sets of S-polynomials of $\mathcal{G}$ in order to obtain generators of the ideal $\mathfrak{A}_J$.

3. Superminimal generators and reduction

In this section we introduce the notion of m-truncation ideal and a new polynomial reduction process, that we call *superminimal reduction*, useful to find a new set of equations to define a marked scheme. From the next section on, we will focus on J-marked schemes with J a strongly stable m-truncation. The reason is twofold: on the one hand, strongly stable m-truncation ideals have a good behavior also from the geometric point of view (Theorem 3.3 and Example 3.4); on the other hand, the superminimal reduction is Noetherian when we take a strongly stable m-truncation ideal (Theorem 3.14), but it is not if we just consider a strongly stable ideal (Example 3.13).

3.1. Truncation strongly stable ideals.

Definition 3.1. Let $J \subseteq S$ be a monomial ideal. We will say that J is an *m-truncation* if J is the truncation of J^{sat} in degree m, that is $J = (J^{\mathrm{sat}})_{\geq m}$.

We observe that an m-truncation ideal J is strongly stable if and only if J^{sat} is and that if J is strongly stable, then $J^{\mathrm{sat}} = \underline{J}$.

The following Lemma highlights some simple features of m-truncation strongly stable ideals, which will turn out to be crucial in the proofs of our main results.

Lemma 3.2. *Let J be a strongly stable m-truncation. Then:*

(i) $B_J \cap B_{\underline{J}} = (B_{\underline{J}})_{\geq m}$.
(ii) $\forall x^\beta \in B_{\underline{J}} \setminus B_J$: $x^\beta x_0^{m-|\beta|} \in B_J$.
(iii) $\forall x^\gamma \in S_{\geq m}$, $\forall t \in \mathbb{N}$: $x^\gamma x_0^t \in J \Leftrightarrow x^\gamma \in J$.
(iv) $\mathcal{N}(J)_{\geq m} = \mathcal{N}(\underline{J})_{\geq m}$.
(v) $\forall\ h \in S_{\geq m}$: *$h$ is J-reduced $\Leftrightarrow$ h is $\underline{J}$-reduced.*
(vi) *If I belongs to $\mathcal{M}\mathrm{f}(J)$, then for every homogeneous polynomial h of degree $\geq m$, J-normal forms modulo I satisfy:* $\mathrm{Nf}(x_0^t \cdot h) = x_0^t \cdot \mathrm{Nf}(h)$.

Proof. Facts (i) and (ii) are straightforward consequences of the definition of m-truncations.

For (iii), we only prove the non trivial part "$\Rightarrow$". If $x^\gamma x_0^t \in J$, then x^γ belongs to $\underline{J}$. Since J is an m-truncation and $x^\gamma \in S_{\geq m}$, then $x^\gamma \in J$ too.

Statements (iv) and (v) are obviously equivalent to (iii).

For (vi), we recall that the J-reduced form modulo I of any polynomial is unique since I belongs to $\mathcal{M}\mathrm{f}(J)$. By (iii), both $\mathrm{Nf}(x_0^t \cdot h)$ and $x_0^t \cdot \mathrm{Nf}(h)$ are J-reduced forms of $x_0^t h$ and then they coincide. □

Theorem 3.3. *Let J be a strongly stable m-truncation ideal. Two different ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{M}\mathrm{f}(J)$ give rise to different subschemes of $\mathbb{P}^n$, thus they correspond to different points of the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ with $p(t)$ the Hilbert polynomial of S/J.*

Proof. By the uniqueness of the reduced form, there is a monomial $x^\alpha \in B_J$ such that the corresponding polynomials $f_\alpha^{\mathfrak{a}}$ and $f_\alpha^{\mathfrak{b}}$ of the J-marked bases of $\mathfrak{a}$ and $\mathfrak{b}$, respectively, are different and moreover such that $f_\alpha^{\mathfrak{a}} \notin \mathfrak{b}$ and $f_\alpha^{\mathfrak{b}} \notin \mathfrak{a}$. If $\mathfrak{a}$ and $\mathfrak{b}$ defined the same projective scheme, we would have $\mathfrak{a}_r = \mathfrak{b}_r$ for some $r \gg 0$. Hence $x_0^{r-m} f_\alpha^{\mathfrak{a}} - x_0^{r-m} f_\beta^{\mathfrak{b}} = x_0^{r-m}(-T(f_\alpha^{\mathfrak{a}}) + T(f_\beta^{\mathfrak{b}}))$ is a non-zero polynomial that belongs (for instance) to $\mathfrak{a}$. Moreover, due to Lemma 3.2, (iii), $x_0^{r-m}(-T(f_\alpha^{\mathfrak{a}}) + T(f_\beta^{\mathfrak{b}}))$ is J-reduced modulo $\mathfrak{a}$: this is impossible because of Proposition 1.11, (iv). □

The following example shows that, if J is a strongly stable ideal but not an m-truncation, different ideals in $\mathcal{M}\mathrm{f}(J)$ may define the same subscheme in $\mathbb{P}^n$. This is the first reason why we will focus mainly on strongly stable m-truncations.

Example 3.4. In the ring $S = K[x_0, x_1, x_2]$, let us consider the strongly stable ideal $J = (x_2, x_1^2, x_1 x_0)$ and for every $c \in K$ the ideal $\mathfrak{a}_c = (x_2 + c x_1, x_1^2, x_1 x_0)$. An easy computation shows that the ideals $\mathfrak{a}_c$ belong to $\mathcal{M}\mathrm{f}(J)$ and are pairwise different. However, x_2^2, x_2x_1, x_2x_0 belong to $\mathfrak{a}_c$: indeed $x_2^2 = (x_2+cx_1)(x_2-cx_1)+c^2x_1^2$, $x_2x_1 = (x_2+cx_1)x_1 - cx_1^2$, $x_2x_0 = (x_2+cx_1)x_0 - cx_1x_0$; hence the saturation of $\mathfrak{a}_c$ is J. Then, the subschemes $\mathrm{Proj}\,(S/\mathfrak{a}_c)$ of $\mathbb{P}^2$ coincide. We can observe that the difference between the ideals $\mathfrak{a}_c$ disappears if we only consider their homogeneous components of degree ≥ 2.

3.2. **Superminimals.** In the following we will use the notation stated in Definition 1.3.

Definition 3.5. Let J be a strongly stable ideal. The *set of superminimal generators* of J is

$$sB_J = \{x^\beta \in B_J \mid x^{\underline{\beta}} \in B_{\underline{J}}\}.$$

Remark 3.6. Another special set of monomials for a strongly stable ideal J is the so-called set of *Borel generators* (see [12]), namely the smallest subset of B_J such that J is the minimum strongly stable ideal containing them. Although there is a clear analogy between the ideas underlying the definition of superminimal generators and that of Borel generators, however they do not coincide in general.

Example 3.7. Consider $\underline{J} := (x_2^3, x_2^2x_1, x_2x_1^2, x_1^6) \subseteq K[x_0,x_1,x_2]$ and its 5-truncation ideal $J := \underline{J}_{\geqslant 5}$. The set of superminimal generators of J is $sB_J = \{x_2^3x_0^2, x_2^2x_1x_0^2, x_2x_1^2x_0^2, x_1^6\}$, while the set of Borel generators of J is $\{x_2x_1^2x_0^2, x_1^6\}$, because $x_2x_1^2x_0^2 \in J$ imposes $x_2^2x_1x_0^2 = \mathrm{e}^+_{1,2}(x_2x_1^2x_0^2) \in J$ and $x_2^3x_0^2 = \mathrm{e}^+_{1,2} \circ \mathrm{e}^+_{1,2}(x_2x_1^2x_0^2) \in J$.

Example 3.8. Consider $J := (x_2^2, x_2x_1^2, x_2x_1x_0, x_2x_0^2) \subseteq K[x_0,x_1,x_2]$ whose saturation is $\underline{J} = (x_2)$. The set of superminimal generators of J is $sB_J = \{x_2x_0^2\}$, while the set of Borel generators of J is $\{x_2^2, x_2x_0^2\}$.

Definition 3.9. Let J be a strongly stable ideal. A finite set of marked polynomials $f_\beta = x^\beta - \sum c_{\beta\gamma}x^\gamma$, with $\mathrm{Ht}(f_\beta) = x^\beta$, is a *$J$-marked superminimal set* if the head terms form the set of superminimal generators sB_J of J, they are pairwise different, and $x^\gamma \in \mathcal{N}(J)$. We call *tail* of f_β the homogeneous polynomial $T(f_\beta) := x^\beta - f_\beta$.

Every J-marked set G contains a (unique) subset sG of this type, that is called *the set of superminimals of G*; if G is a J-marked basis, sG is called *J-superminimal basis*.

Remark 3.10. If Γ is a J-marked superminimal set, it can always be completed to a (non-unique) J-marked set G. For instance $G = \Gamma \cup (B_J \setminus sB_J)$.

On the other hand, if $I \in \mathcal{M}\mathrm{f}(J)$, then its J-superminimal basis is the only J-marked superminimal set contained in I. In fact, for every $x^\beta \in sB_J$, if f_β belongs to both I and a J-marked superminimal set, then $x^\beta - f_\beta$ has to be a J-reduced form of x^β modulo I, which is the unique normal form $\mathrm{Nf}(x^\beta)$ modulo I.

Definition 3.11. Consider a strongly stable ideal J, a J-marked set G and two polynomials h and h_1. We say that h is in *sG_*-relation* with h_1 if there is a monomial $x^\gamma \in \mathrm{Supp}(h) \cap J$, $c = \mathrm{Coeff}(x^\gamma)$, such that x^γ is divisible by a superminimal generator x^α of J, *with* $x^\gamma = x^{\underline{\alpha}} *_{\underline{J}} x^\eta = x^\alpha \cdot x^\epsilon$ and $h_1 = h - c\cdot x^\epsilon f_\alpha$, that is h_1 is obtained by replacing the monomial x^γ in h by $x^\epsilon \cdot T(f_\alpha)$. We call *superminimal reduction* the transitive closure of the above relation and denote it by $\xrightarrow{sG*}$. Moreover, we say that:

- *h can be reduced to* h_1 by $\xrightarrow{sG*}$ if $h \xrightarrow{sG*} h_1$;
- *h is non-reducible w.r.t.* $\xrightarrow{sG*}$ if no step of reduction on h by $\xrightarrow{sG*}$ can be performed;
- *h is strongly reduced* if for every t, $x_0^t \cdot h$ is non-reducible w.r.t. $\xrightarrow{sG*}$.

Remark 3.12.

(i) We use the notation $\xrightarrow{sG*}$ to underline that this reduction also involves the decomposition $*_{\underline{J}}$ of Definition 2.4 and it is not the usual polynomial reduction w.r.t. a set of marked polynomials sG. Indeed even if a polynomial h is non-reducible w.r.t. $\xrightarrow{sG*}$, its support can contain some monomial which is multiple of a monomial in sB_J (see Example 3.15); hence h would be reducible w.r.t. the usual reduction $\xrightarrow{sG}$.

(ii) A homogeneous polynomial h is strongly reduced if and only if no monomial in $\mathrm{Supp}(h)$ is divisible by a monomial of $B_{\underline{J}}$, that is h is $\underline{J}$-reduced. In fact, if $x^\gamma \in \mathrm{Supp}(h)\cap \underline{J}$ then $x^\gamma = x^{\underline{\alpha}} *_{\underline{J}} x^\eta$ and there is t such that $x^\alpha = x_0^t x^{\underline{\alpha}} \in B_J$. Thus $x_0^t h$ can be reduced by $\xrightarrow{sG*}$ using the polynomial f_α.

(iii) The polynomials $x^\epsilon f_\alpha$ that we use for the reduction procedure $\xrightarrow{sG*}$ have head terms pairwise different. Moreover, if $x^\delta f_{\alpha'}$ is used in the $\xrightarrow{sG*}$ reduction of $x^\epsilon T(f_\alpha)$ then $x^{\underline{\delta}} <_{\mathtt{Lex}} x^{\underline{\epsilon}}$.

If we consider a strongly stable ideal J with no further hypothesis, we cannot generalize the properties of the reduction $\xrightarrow{V_\ell}$ to $\xrightarrow{sG*}$, as shown in the following example.

Example 3.13. In the ring $S = K[x_0, x_1, x_2]$ (with $x_2 > x_1 > x_0$) let us consider the strongly stable ideal $J = (x_2^3, x_2^2x_1, x_2x_1^2, x_2^2x_0, x_2x_1x_0, x_1^4, x_1^3x_0, x_1^2x_0^2)$ and its saturation $\underline{J} = (x_2^2, x_2x_1, x_1^2)$. The set of superminimals of J is $sB_J = \{x_2^2x_0, x_2x_1x_0, x_1^2x_0^2\}$. Let us consider the J-marked superminimal set $sG = \{f_{x_2^2x_0} = x_2^2x_0,\ f_{x_2x_1x_0} = x_2x_1x_0 - x_1^3,\ f_{x_1^2x_0^2} = x_1^2x_0^2 - x_2x_0^3\}$. The superminimal reduction w.r.t. sG is not Noetherian. For instance:

$$x_1^3x_0^2 \xrightarrow{sG*} T(f_{x_1^2x_0^2})\cdot x_1 = x_2x_0^3x_1 \xrightarrow{sG*} T(f_{x_2x_1x_0})\cdot x_0^2 = x_1^3x_0^2.$$

However, if we assume that the strongly stable ideal J is also an m-truncation ideal, then the reduction $\xrightarrow{sG*}$ turns out to be Noetherian and satisfies several good properties, similar to the ones of $\xrightarrow{V_\ell}$.

Theorem 3.14. *Let J be a strongly stable m-truncation ideal and sG be a J-marked superminimal set. Then:*

(i) $\xrightarrow{sG*}$ *is Noetherian.*

(ii) For every homogeneous polynomial h there exist t and a unique polynomial $h(t)$ strongly reduced such that $x_0^t \cdot h \xrightarrow{sG*} h(t)$. If $\bar{t}$ is the minimum one and $\bar{h} := h(\bar{t})$, then $h(t) = x_0^{t-\bar{t}}\cdot \bar{h}$ for every $t \geq \bar{t}$. There is an effective procedure that computes $\bar{t}$ and $\bar{h}$.

If moreover sG is the superminimal basis of an ideal I of $\mathcal{M}\mathrm{f}(J)$, then:

(iii) $\xrightarrow{sG*}$ *computes the J-normal forms modulo I. More precisely, for every homogeneous polynomial h:*

$$\mathrm{Nf}(h) = \begin{cases} h, & \text{if } \deg(h) < m \\ \bar{h}/x_0^{\bar{t}}, & \text{if } \deg(h) \geq m \text{ and } x_0^{\bar{t}}\cdot h \xrightarrow{sG*} \bar{h} \end{cases}$$

(iv) $\xrightarrow{sG*}$ *solves the ideal-membership problem for I: for every homogeneous polynomial h:*

$$h \in I \Leftrightarrow \deg(h) \geq m \text{ and } x_0^{\bar{t}}\cdot h \xrightarrow{sG*} 0$$

(v) There is a one-to-one correspondence between ideals in $\mathcal{M}\mathrm{f}(J)$ and J-superminimal bases.

Proof.

(i) Since J is a strongly stable and m-truncation ideal, then $\mathcal{N}(J)_{\geq m} = \mathcal{N}(\underline{J})_{\geq m}$ (Lemma 3.2, (iv)). If $\xrightarrow{sG*}$ was not Noetherian, by Lemma 2.5 applied to $\underline{J}$, we would be able to find infinite descending chains of monomials w.r.t. $<_{\mathtt{Lex}}$.

(ii) It is sufficient to prove the thesis for monomials x^γ in J. Let $x^\gamma = x^{\underline{\alpha}} *_{\underline{J}} x^\eta$. If $x^\eta = 1$, then $x^\alpha = x_0^{t_\alpha} \cdot x^{\underline{\alpha}}$ is in sB_J, f_α belongs to sG and $x_0^{t_\alpha} \cdot x^{\underline{\alpha}} \xrightarrow{sG*} T(f_\alpha)$, where $\mathrm{Supp}(T(f_\alpha)) \subseteq \mathcal{N}(J)$. In this case $\overline{h} = T(f_\alpha)$ and $\overline{t} = t_\alpha$. If $x^\eta \neq 1$, we can assume that the thesis holds for any monomial $x^{\gamma'} = x^{\underline{\beta}} *_{\underline{J}} x^{\eta'}$, such that $x^{\eta'} <_{\mathtt{Lex}} x^\eta$.

We perform a first reduction $x_0^{t_\alpha} \cdot x^\gamma \xrightarrow{sG*} x^\eta \cdot T(f_\alpha)$. If $x^\eta \cdot T(f_\alpha)$ is strongly reduced, we are done. Otherwise, we have $x^\eta \neq x_0^{|\eta|}$. For every monomial $x^{\gamma'} \in \mathrm{Supp}(x^\eta \cdot T(f_\alpha)) \cap \underline{J}$ we have $x^{\gamma'} = x^{\underline{\beta}'} *_{\underline{J}} x^{\eta'}$, with $x^{\eta'} <_{\mathtt{Lex}} x^\eta$ by Lemma 2.5. So, we have also $x_0^t \cdot x^{\eta'} <_{\mathtt{Lex}} x^\eta$, for every t. By the inductive hypothesis we can find a suitable power t of x_0 such that every monomial in $x_0^t \cdot x^\eta \cdot T(f_\alpha)$ can be reduced by $\xrightarrow{sG*}$ to a strongly reduced polynomial.

It remains to prove the uniqueness of the strongly reduced polynomial $h(t)$. Let us consider two different strongly reduced $\xrightarrow{sG*}$ reductions of $x_0^t h$: their difference is again strongly reduced and can be written as $\Sigma a_i x^{\eta_i} f_{\alpha_i}$ with $a_i \in K$, $a_i \neq 0$ and $x^{\eta_i} f_{\alpha_i}$ pairwise different. Let $x^{\eta_1} f_{\alpha_1}$ be such that for every $i \geq 2$, either $x^{\eta_1} >_{\mathtt{Lex}} x^{\eta_i}$ or $x^{\eta_1} = x^{\eta_i}$ and $x^{\alpha_1} >_{\mathtt{Lex}} x^{\alpha_i}$. Then $x^{\eta_1} x^{\alpha_1}$ should cancel with a monomial in $\mathrm{Supp}(x^{\eta_i} T(f_{\alpha_i}))$ for some i, but this is impossible as observed in Remark 3.12, (iii).

Observe that, though for a fixed $x^\gamma = x^{\underline{\alpha}} *_J x^\eta$, there are infinitely many monomials $x^{\gamma'} = x^{\underline{\beta}} *_J x^{\eta'}$ such that $x^{\eta'} <_{\mathtt{Lex}} x^\eta$, we use the inductive hypothesis only with respect to the finite number of them that appear on the support of $\cdot x^\eta \cdot T(f_\alpha)$. For this reason our procedure is effective.

From now on we consider $I \in \mathcal{M}\mathrm{f}(J)$; therefore if h is a homogeneous polynomial and $h \xrightarrow{sG*} h_1$ with h_1 strongly reduced, then by uniqueness of J-normal forms modulo I we have $h_1 = \mathrm{Nf}(h)$.

(iii) If $\deg h < m$ we are done. Otherwise from (ii) we have that $x_0^{\bar{t}} \cdot h \xrightarrow{sG*} \overline{h}$ and $\bar{h}$ is a J-reduced form modulo I. Thus $x_0^t \cdot \mathrm{Nf}(h)$ is J-reduced too (Lemma 3.2, (iii)) and we get the desired equality by uniqueness of J-normal forms modulo I.
(iv) This is a consequence of (iii) and of Proposition 1.11 (iv).
(v) This is the straightforward consequence of (iv). □

Whenever J is a strongly stable m-truncation ideal and sG is the superminimal basis of an ideal $I \in \mathcal{M}\mathrm{f}(J)$, then sG is a subset of the set V of Definition 2.1. Nevertheless, it is interesting to notice that not every step of reduction by $\xrightarrow{sG*}$ is also a step of reduction by $\xrightarrow{V_\ell}$, as shown in the following example.

Example 3.15. Consider $J = (x_1^2, x_0x_2, x_1x_2, x_2^2)$ which is a strongly stable ideal and a 2-truncation of $\underline{J} = (x_2, x_1^2)$ in $K[x_0, x_1, x_2]$. Let G be a J-marked set.

- The monomial $x_2 \cdot x_1^2$ is non-reducible w.r.t. sG, because the only monomial of sB_J dividing it is x_1^2, but $x_2 x_1^2 = x_2 *_{\underline{J}} x_1^2$. On the other hand, $x_2 x_1^2 = x_2 x_1 *_J x_1$, so $x_2 x_1^2 \xrightarrow{V_3} x_1 T(f)$ where $f \in V_2$, $\mathrm{Ht}(f) = x_2 x_1$.
- The only way to reduce $x_0 \cdot x_2^2$ via $\xrightarrow{V_3}$ leads to $x_0 \cdot T(f')$, where f' is the unique polynomial of V_2 such that $\mathrm{Ht}(f') = x_2^2$. Moreover, $x_0 \cdot T(f')$ is not further reducible, because all the monomials of its support belong to $\mathcal{N}(J)$. On the other hand, according to Definition 3.11, a first step of reduction of the monomial $x_0 \cdot x_2^2$ via $\xrightarrow{sG*}$ is $x_0 x_2^2 \xrightarrow{sG*} x_2 \cdot T(f'')$, where f'' is the polynomial in sG with $\mathrm{Ht}(f'') = x_0 \cdot x_2$. Since x_2 is a monomial of $B_{\underline{J}}$, every monomial appearing in $\mathrm{Supp}(x_2 \cdot T(f''))$ belongs to J, and so we will need further steps of reduction via $\xrightarrow{sG*}$ to compute a polynomial non-reducible w.r.t. sG.

4. Buchberger-like criterion by superminimal reduction

In the present and following sections, we assume that $J \subseteq S$ *is a strongly stable* m*-truncation ideal*, in order to apply the main results of Section 3, mainly those concerning the new reduction process $\xrightarrow{sG*}$ (Theorem 3.14). We will also use the sets of polynomials V and W which are defined from a J-marked set G (see Definition 2.1), and the reduction relation $\xrightarrow{V_\ell}$.

In Section 2, we proved that J-marked bases are characterized by a Buchberger-like criterion on the reduction of S-polynomials between elements of G by $\xrightarrow{V_\ell}$ (Theorem 2.11). Afterwards, in Section 3 we showed that every homogeneous ideal I in $\mathcal{M}\mathrm{f}(J)$ is completely determined by its superminimal basis sG and that J-normal forms modulo I can be computed using $\xrightarrow{sG*}$, that is again using polynomials in the subset sG of G (Theorem 3.14).

Therefore, it is natural to ask whether one can obtain a Buchberger-like criterion only considering S-polynomials among elements in sG. Unfortunately, the answer is negative, as clearly shown by Example 4.1. However, we can prove a few variants of the Buchberger-like criterion of Theorem 2.11, in which the set of superminimals sG and the superminimal reduction process $\xrightarrow{sG*}$ replace G and $\xrightarrow{V_\ell}$. Using these new criteria in the next section we will be able to obtain sets of equations defining $\mathcal{M}\mathrm{f}(J)$ in a smaller set of variables than those of Subsection 2.3.

Example 4.1. We consider the strongly stable 2-truncation ideal

$$J = (x_3^2, x_3x_2, x_3x_1, x_3x_0, x_2^2) \subseteq K[x_0, x_1, x_2, x_3]$$

whose saturation is $\underline{J} = (x_3, x_2^2)$. In this case, sB_J contains only two monomials, x_3x_0 and x_2^2. If G is any J-marked set, then $sG = \{f_{x_3x_0}, f_{x_2^2}\}$. The unique S-polynomial among superminimal elements is

$$S(f_{x_3x_0}, f_{x_2^2}) = x_2^2 f_{x_3x_0} - x_3x_0 f_{x_2^2} = x_3x_0 \cdot T(f_{x_2^2}) - x_2^2 \cdot T(f_{x_3x_0}).$$

Any monomial appearing in $\mathrm{Supp}(T(f_{x_3x_0}))$ is in $\mathcal{N}(J)_2 = K[x_0, x_1, x_2]_2 \setminus \{x_2^2\}$. Then any monomial appearing in $\mathrm{Supp}(x_2^2 \cdot T(f_{x_3x_0}))$ is further reduced by $f_{x_2^2}$, obtaining by $\xrightarrow{V_4}$ or $\xrightarrow{sG*}$

$$S(f_{x_3x_0}, f_{x_2^2}) = x_3x_0 \cdot T(f_{x_2^2}) - T(f_{x_2^2}) \cdot T(f_{x_3x_0}) = T(f_{x_2^2}) \cdot f_{x_3x_0} \to 0.$$

Nevertheless, even if the only S-polynomial among superminimal generators reduces to 0, if we consider $sG = \{f_{x_3x_0}, f_{x_2^2}\}$ with $f_{x_3x_0} = x_3x_0 + x_1^2$ and $f_{x_2^2} = x_2^2$, then for any choice of $f_{x_3^2}, f_{x_3x_2}, f_{x_3x_1}$, the S-polynomial among $f_{x_3x_1}$ and $f_{x_3x_0}$ does not reduce to 0:

$$S(f_{x_3x_1}, f_{x_3x_0}) = x_0 f_{x_3x_1} - x_1 f_{x_3x_0} = \sum_{x^{\alpha_i} \in \mathcal{N}(J)_2} a_i x^{\alpha_i} x_0 - x_1^3.$$

The monomials $x^{\alpha_i}x_0$ are in $\mathcal{N}(J)_3$ and are strongly reduced. Furthermore, x_1^3 does not appear among monomials $x^{\alpha_i}x_0$, so it is not canceled, and it is strongly reduced too. Therefore, for any choice of coefficients in the tail of $f_{x_3x_1}$, we have an S-polynomial which is not reducible to 0, and any J-marked set containing $f_{x_3x_0} = x_3x_0 + x_1^2$ is not a J-marked basis.

4.1. Buchberger-like criteria via $\xrightarrow{sG*}$: first variant.

In this subsection we prove that the Buchberger-like criterion of Theorem 2.11 and Corollary 2.13 can be rephrased in terms of the reduction process $\xrightarrow{sG*}$. The involved S-polynomials will be all those between elements in G (Theorem 4.5), or only EK-polynomials between elements of G (Corollary 4.6). We will need a few lemmas.

Lemma 4.2. *Let J be a strongly stable m-truncation ideal, G be a J-marked set and h be a homogeneous polynomial of degree $\ell \geq m$. Then:*

$$h \in \langle V_\ell \rangle \Leftrightarrow x_0 \cdot h \in \langle V_{\ell+1} \rangle.$$

Proof. If $h \in \langle V_\ell \rangle$, then $x_0 \cdot h \in \langle V_{\ell+1} \rangle$ by definition of V.

Vice versa, assume that $x_0 \cdot h \in \langle V_{\ell+1} \rangle$. This is equivalent to $x_0 \cdot h \xrightarrow{V_{\ell+1}} 0$. Every monomial in $\mathrm{Supp}(x_0 \cdot h)$ can be written as $x_0 \cdot x^\epsilon$; observe that $x_0 \cdot x^\epsilon \notin B_J$, because $\deg(x_0 \cdot x^\epsilon) > m$, by Lemma 3.2, (i). Then, if $x_0 \cdot x^\epsilon$ belongs to J, we can decompose it as $x_0 \cdot x^\epsilon = x^\alpha *_J x^\eta$, $x^\alpha \in B_J$ and $x^\eta \neq 1$. Since $\min(x^\alpha) \geq \max(x^\eta)$, we have that x^η is divisible by x_0. So $x^\eta = x_0 \cdot x^{\eta'}$.

Summing up, in order to reduce the monomial $x_0 \cdot x^\epsilon$ of $\mathrm{Supp}(x_0 \cdot h)$ using V, we use the polynomial $x_0 \cdot x^{\eta'} \cdot f_\alpha \in V$, $\mathrm{Ht}(f_\alpha) = x^\alpha$. If the coefficient of $x_0 \cdot x^\epsilon$ in $x_0 \cdot h$ is a, we obtain

$$x_0 \cdot h \xrightarrow{V_{\ell+1}} x_0 \cdot (h - a \cdot x^{\eta'} f_\alpha).$$

At every step of reduction, we obtain a polynomial which is divisible by x_0. In particular,

$$x_0 \cdot h \in \langle V_{\ell+1} \rangle \Rightarrow x_0 \cdot h = x_0 \cdot \sum a_i x^{\eta_i} f_{\alpha_i}, \text{ where } x_0 \cdot x^{\eta_i} f_{\alpha_i} \in V_{\ell+1}.$$

Then we have that $h = \sum a_i x^{\eta_i} f_{\alpha_i}$ and $x^{\eta_i} f_{\alpha_i} \in V_\ell$, that is $h \in \langle V_\ell \rangle$. □

Consider $f_\alpha, f_{\alpha'} \in G$, the S-polynomial $S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}$ and assume that $x^{\gamma'} <_{\mathtt{Lex}} x^\gamma$. By Lemma 2.8, (iii), if $S(f_\alpha, f'_\alpha) \xrightarrow{V_\ell} h$, then $S(f_\alpha, f'_\alpha) - h = \sum a_j x^{\delta_j} f_{\beta_j}$ with $x^{\delta_j} f_{\beta_j} \in V_\ell$, $x^{\delta_j} <_{\mathtt{Lex}} x^\gamma$. Now we show that a similar result holds for the superminimal reduction $\xrightarrow{sG*}$.

Lemma 4.3. *Let J be a strongly stable m-truncation ideal, G be a J-marked set and $f_\alpha, f_{\alpha'}$ be two polynomials belonging to G. Consider the S-polynomial $S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}$, with $x^{\gamma'} <_{\mathtt{Lex}} x^\gamma$. If $x_0^t \cdot S(f_\alpha, f'_\alpha) \xrightarrow{sG*} h$, then $x_0^t \cdot S(f_\alpha, f'_\alpha) - h = \sum a_j x^{\eta_j} f_{\beta_j}$ with $f_{\beta_j} \in sG$, $x^{\eta_j} <_{\mathtt{Lex}} x^\gamma$ and $x^{\underline{\eta_j}} <_{\mathtt{Lex}} x^{\underline{\gamma}}$.*

Proof. Every monomial $x_0^t \cdot x^\gamma \cdot x^\epsilon$ in $\mathrm{Supp}(x_0^t \cdot x^\gamma \cdot T(f_\alpha)) \cap J$ decomposes as $x_0^t \cdot x^\gamma \cdot x^\epsilon = x^{\underline{\beta}} *_{\underline{J}} x^\eta$, $x^\eta <_{\mathtt{Lex}} x_0^t x^\gamma$ and $x^{\underline{\eta}} <_{\mathtt{Lex}} x^{\underline{\gamma}}$ by Lemma 3.2, (iv) and Lemma 2.5. The same holds for any further reduction and the same argument applies to monomials appearing in $\mathrm{Supp}(x_0^t \cdot x^{\gamma'} \cdot T(f_{\alpha'}))$. □

We point out that Lemma 4.3 does not hold without the hypothesis that J is an m-truncation ideal, as shown by the following example.

Example 4.4. In $S = K[x_0, x_1, x_2, x_3]$, consider the strongly stable ideal

$$J = (x_3^2, x_3 x_2, x_3 x_1)_{\geq 4} + (x_2^2)_{\geq 6},$$

whose saturation is $\underline{J} = (x_3^2, x_2 x_3, x_1 x_3, x_2^2)$. J is not an m-truncation for any m. Consider a J-marked set G and $f_\alpha, f_\beta \in G$ such that $\mathrm{Ht}(f_\alpha) = x_0^2 x_2 x_3$ and $\mathrm{Ht}(f_\beta) = x_0^2 x_1 x_3$ and consider $x_2^4 \in \mathrm{Supp}(T(f_\beta))$. Then $S(f_\alpha, f_\beta) = x_1 f_\alpha - x_2 f_\beta$. If we apply Definition 3.11, we reduce $x_2^4 \in \mathrm{Supp}(S(f_\alpha, f_\beta))$ by $\xrightarrow{sG*}$, pre-multiplying by x_0^4. We get that $x_0^4 x_2^4$ belongs to $\mathrm{Supp}(x_0^4 S(f_\alpha, f_\beta))$ and $x_0^4 x_2^4 = x_2^2 *_{\underline{J}} x_0^4 x_2^2$. But $x_2^2 >_{\mathtt{Lex}} x_2$.

Theorem 4.5. *Let J be a strongly stable m-truncation ideal, G be a J-marked set and I be the homogeneous ideal generated by G. The followings are equivalent:*

(i) $I \in \mathcal{M}\mathrm{f}(J)$;

(ii) $\forall f_\alpha, f_{\alpha'} \in G$, $\exists\, t$ *such that* $x_0^t \cdot S(f_\alpha, f_{\alpha'}) \xrightarrow{sG*} 0$;

(iii) $\forall f_\alpha, f_{\alpha'} \in G$, $\exists\, t$ *such that* $x_0^t \cdot S(f_\alpha, f_{\alpha'}) = x_0^t (x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}) = \sum a_j x^{\eta_j} f_{\alpha_j}$, *with* $x^{\eta_j} <_{\mathtt{Lex}} \max_{\mathtt{Lex}}\{x^\gamma, x^{\gamma'}\}$ *and* $f_{\alpha_j} \in sG$.

Proof. If $I \in \mathcal{M}\mathrm{f}(J)$, we can apply Theorem 3.14, (iv) because any S-polynomial among elements in G belongs to I.

If statement (ii) holds, then we get (iii) by Lemma 4.3.

We now assume that statement (iii) holds and by Lemma 2.2 it is sufficient to prove that $\langle V\rangle = \langle W\rangle$ using an argument analogous to that applied in the proof of Theorem 2.11. It is sufficient to prove that $x^\eta \cdot V \subseteq \langle V\rangle$, for every monomial x^η. We proceed by induction on the monomials x^η, ordered according to $>_{\texttt{Lex}}$. The thesis is obviously true for $x^\eta = 1$. We then assume that the thesis holds for any monomial $x^{\eta'}$ such that $x^{\eta'} <_{\texttt{Lex}} x^\eta$.

If $|\eta| > 1$, we can consider any product $x^\eta = x^{\eta_1}\cdot x^{\eta_2}$, x^{η_1} and x^{η_2} non-constant. Since $x^{\eta_i} <_{\texttt{Lex}} x^\eta, i = 1, 2$, we immediately obtain by induction

$$x^\eta \cdot V = x^{\eta_1}\cdot (x^{\eta_2}\cdot V) \subseteq x^{\eta_1}\langle V\rangle \subseteq \langle V\rangle.$$

If $|\eta| = 1$, then we need to prove that $x_i \cdot V \subseteq \langle V\rangle$. Since $x_0 V \subseteq V$, it is then sufficient to prove the thesis for $x^\eta = x_i$, $i \geq 1$, assuming that the thesis holds for every $x^{\eta'} <_{\texttt{Lex}} x_i$. We consider $g_\beta = x^\delta f_\alpha \in V$, where $\max(x^\delta) \leq \min(x^\alpha)$. If $x_i g_\beta$ does not belong to V, then $\max(x_i\cdot x^\delta) > \min(x^\alpha)$, so $x_i > \min(x^\alpha)$ because $\max(x^\delta) \leq \min(x^\alpha)$ by construction. In particular, $x_i > \min(x^\alpha) \geq \max(x^\delta)$, so $x_i >_{\texttt{Lex}} x^\delta$ and it is sufficient to prove the thesis for $x_i f_\alpha$.

We consider an S-polynomial $S(f_\alpha, f_{\alpha'}) = x_i f_\alpha - x^\gamma f_{\alpha'}$ such that $x^\gamma <_{\texttt{Lex}} x_i$. Such S-polynomial always exists: for instance, we can consider $x_i x^\alpha = x^{\alpha'} *_J x^{\eta'}$.

By hypothesis there is t such that $x_0^t S(f_\alpha, f_{\alpha'}) = x_0^t(x_i f_\alpha - x^{\eta'} f_{\alpha'}) = \sum a_j x^{{\eta'}_j} f_{\alpha_j}$ where $x_0^t x^{\eta'}, x^{{\eta'}_j}$ are lower than x_i w.r.t. $\texttt{Lex}$. Then $x_0^t x^{\eta'} f_{\alpha'}$, $x^{{\eta'}_j} f_{\alpha_j}$ belong to $\langle V\rangle$ by induction and we conclude that $x_i f_\alpha \in \langle V\rangle$, by Lemma 4.2. □

The previous theorem is the analogous of Theorem 2.11 for the reduction process $\xrightarrow{sG*}$. As stated in Corollary 2.13 concerning the Buchberger-like criterion for the reduction $\xrightarrow{V_\ell}$, also in Theorem 4.5 it would be sufficient to assume statement (iii) only for EK-polynomials.

Corollary 4.6. *With the same notations of Theorem 4.5, the followings are equivalent:*

(i) $I \in \mathcal{M}\mathrm{f}(J)$

(ii) for every EK-polynomial between elements of G, $\exists\, t :\, x_0^t S^{EK}(f_\alpha, f_{\alpha'}) \xrightarrow{sG*} 0$.

(iii) for every EK-polynomial between elements of G, $\exists\, t$ such that $x_0^t \cdot S^{EK}(f_\alpha, f_{\alpha'}) = x_0^t(x_i f_\alpha - x^{\gamma'} f_{\alpha'}) = \sum a_j x^{\eta_j} f_{\alpha_j}$, with $x^{\eta_j} <_{\texttt{Lex}} x_i$ and $f_{\alpha_j} \in sG$.

4.2. **Buchberger-like criteria via $\xrightarrow{sG*}$: second variant.** As before, let J be a strongly stable m-truncation ideal. By Example 4.1, we have already shown that reductions of S-polynomials between elements of sG are not sufficient to characterize ideals of $\mathcal{M}\mathrm{f}(J)$; hence some more conditions are necessary. To this aim, we add some further S-polynomials.

Indeed, Theorem 4.7 uses the set L_1 of some couples of polynomials of sG and the set L_2 of some particular couples of elements of G of minimal degree m to obtain a new characterization of $\mathcal{M}\mathrm{f}(J)$. Actually the elements of L_1 are not all the possible couples of elements in sG, but a subset of them, corresponding to a minimal set of generators for the first module of syzygies of the Eliahou-Kervaire resolution of $\underline{J}$.

Theorem 4.7. *Consider a strongly stable m-truncation ideal J and G a J-marked set. Let us define the following sets:*

$$L_1 := \left\{ (f_\alpha, f_{\alpha'}) \mid f_\alpha, f_{\alpha'} \in sG \text{ and } x_i x^{\underline{\alpha}} = x^{\underline{\alpha}'} *_{\underline{J}} x^\eta \right\},$$

$$L_2 := \left\{ (f_\alpha, f_{\alpha'}) \mid f_\alpha, f_{\alpha'} \in G_m \text{ and } x_i x^{\alpha'} = x_0 x^\alpha,\ x_i = \min_{j>0}\{x_j : x_j \mid x^\alpha\} \right\}.$$

Then:

$$I \in \mathcal{M}\mathrm{f}(J) \Leftrightarrow\ \forall\ (f_\alpha, f_{\alpha'}) \in L_1 \cup L_2,\ \exists\, t \text{ such that } x_0^t \cdot S(f_\alpha, f_{\alpha'}) \xrightarrow{sG*} 0.$$

Proof. If I belongs to $\mathcal{M}\mathrm{f}(J)$, then it is enough to apply Theorem 4.5, (ii).

Vice versa, by Lemma 2.2 it is sufficient to prove that $\langle V\rangle = \langle W\rangle$, that is $x_i \cdot V \subseteq \langle V\rangle$ for every $i = 0,\dots,n$. We proceed by induction on the variables. By construction we have $x_0 \cdot V \subseteq \langle V\rangle$. We now assume that $(x_0,\dots,x_{i-1})V \subseteq \langle V\rangle$ and we prove that $x_i \cdot V \subseteq \langle V\rangle$. Consider $x^\delta f_\beta \in V$. The thesis is that $x_i \cdot x^\delta f_\beta$ is contained in $\langle V\rangle$. If $x_i x^\delta f_\beta$ does not belong to V, then $\max(x_i \cdot x^\delta) > \min(x^\beta)$, so $x_i > \min(x^\beta)$ because $\max(x^\delta) \leq \min(x^\beta)$ by construction. In particular, $x_i > \min(x^\beta) \geq \max(x^\delta)$, so that it is sufficient to prove the thesis for $x_i f_\beta$, because by induction then we have $x^\delta x_i f_\beta \in \langle V\rangle$. Consider $x^\beta = x^{\underline{\alpha}} *_{\underline{J}} x^\eta$.

We have a first case when $x^\eta = 1$. Then $x^\beta = x^{\underline{\alpha}}$ and f_β belongs to sG. We consider $x^{\underline{\alpha}} x_i = x^{\underline{\alpha}'} *_{\underline{J}} x^{\eta'}$. Observe that since $x_i > \min(x^{\underline{\alpha}})$ then x_i does not divide $x^{\eta'}$ and $\max(x^{\eta'}) < x_i$. Consider $x^{\alpha'} = x^{\underline{\alpha}'} \cdot x_0^{t_{\alpha'}}$, so that we can take the polynomial $f_{\alpha'} \in sG$. The pair $(f_\beta, f_{\alpha'})$ belongs to L_1, hence, by the hypothesis and by Lemma 4.3, there is t such that

$$x_0^t S(f_\beta, f_{\alpha'}) = x_0^t(x_0^{t_{\alpha'}} x_i f_\beta - x^{\eta'} f_{\alpha'}) = \sum a_j x^{\eta_j} f_{\alpha_j},$$

with $x^{\eta_j} <_{\mathtt{Lex}} x_i$ and $f_{\alpha_j} \in sG$. Hence we obtain that both $x^{\eta_j} f_{\alpha_j}$ and $x^{\eta'} f_{\alpha'}$ belong to $\langle V\rangle$ by induction on the variables, and so $x_i f_\beta$ belongs to $\langle V\rangle$ (by Lemma 4.2).

We have a second case when $x^\eta = x_0^t$, $t > 0$. Then, $|\beta| = m$ and f_β belongs to sG. Let $x_i x^\beta = x^{\underline{\alpha}'} *_{\underline{J}} x^{\eta'}$. If $x_i > \min(x^{\underline{\alpha}'})$, then $x^{\eta'}$ is not divisible by x_i and we repeat the argument above. Otherwise, $x_i \leq \min(x^{\underline{\alpha}'})$ and x_i does not divide $x^{\eta'}$, so that $x_i = \min(x^{\underline{\alpha}'})$ and $x^{\eta'} <_{\mathtt{Lex}} x_i$. Then, we take $x^{\beta'} = \frac{x^\beta}{x_0} \cdot x_i$ that belongs to B_J because it has degree m. The pair $(f_\beta, f_{\beta'})$ belongs to L_2 and we repeat the same reasoning above.

We now assume the thesis holds for every $f_{\beta'}$ such that $x^{\beta'} = x^{\underline{\alpha}'} *_{\underline{J}} x^{\eta'}$ with $x^{\eta'} <_{\mathtt{Lex}} x^\eta$. By the base of the induction, we can suppose that $x^\eta \geq_{\mathtt{Lex}} x_1$; so, f_β does not belong to sG and it has degree m. Let $x_j := \min_{l>0}\{x_l \ : \ x_l \mid x^\beta\}$.

Observe that if x_0 does not divide x^β, then $x_j = \min(x^\beta)$; in this case, we have $x_i > x_j$ because $x_i > \min(x^\beta)$. Anyway, first we suppose that $x_i \leq x_j$; $x_j > \min(x^\beta)$ and x_0 divides x^β. We consider $x^{\beta'} = \frac{x^\beta}{x_0} \cdot x_i$, the pair $(f_\beta, f_{\beta'})$ that belongs to L_2 and we repeat the argument of the previous case.

We now assume that $x_i > x_j$ and consider $x^{\beta'} = \frac{x^\beta}{x_j} \cdot x_0 = x^{\underline{\alpha}'} *_{\underline{J}} x^{\eta'}$. Observe that $x^{\eta'} <_{\mathtt{Lex}} x^\eta$ as $x^{\eta'} = \frac{x^\eta}{x_j} \cdot x_0$. Therefore the pair $(f_{\beta'}, f_\beta)$ belongs to L_2; by the hypothesis and by Lemma 4.3, there is an integer t such that

$$x_0^t S(f_{\beta'}, f_\beta) = x_0^t(x_j f_{\beta'} - x_0 f_\beta) = \sum a_l x^{\eta_l} f_{\alpha_l} \tag{3}$$

with $x^{\eta_l} <_{\mathtt{Lex}} x_j$, $f_{\alpha_l} \in sG$. We now multiply (3) by x_i. We observe that $x_i f_{\alpha_l}$ belongs to $\langle V\rangle$, because $f_{\alpha_l} \in sG$ and by the first two cases. Also $x_i f_{\beta'}$ belongs to $\langle V\rangle$ because $x^{\eta'} <_{\mathtt{Lex}} x^\eta <_{\mathtt{Lex}} x_i$. Moreover, $x_j x_i f_{\beta'}$ belongs to $\langle V\rangle$ by induction on the variables. Finally $x_i f_\beta$ belongs to $\langle V\rangle$ thanks to Lemma 4.2. ∎

5. Embedding of $\mathcal{M}\mathrm{f}(J)$ in affine linear spaces of low dimension

In this section we continue to consider a strongly stable m-truncation ideal $J = \underline{J}_{\geq m}$ and, as in Subsection 2.3, we work again with J-marked sets $\mathcal{G}$ where the coefficients of the monomials in the tails are considered as parameters.

Definition 5.1. If $\mathcal{G}$ is the set of marked polynomials given as in (1) for the ideal J, we will call *set of superminimals*, and denote it by $s\mathcal{G}$, the subset of $\mathcal{G}$ made up of $F_\alpha \in \mathcal{G}$ with $\mathrm{Ht}\,(F_\alpha) \in sB_J$. We will denote by C the set of variables appearing in the tails of the polynomials in $\mathcal{G}$ and by $\widetilde{C}$ the set of variables appearing in the tails of the polynomials in $s\mathcal{G}$. $\mathfrak{A}_J$ is the ideal defining the affine subscheme $\mathcal{M}\mathrm{f}(J)$ in the ring $K[C]$.

Observe that the J-marked basis G of every $I \in \mathcal{M}\mathrm{f}(J)$ is obtained by specializing in a suitable way the variables C in $\mathcal{G}$ and that the set of superminimals sG of I is obtained in the same way by $s\mathcal{G}$ through the same specialization of the variables $\widetilde{C}$.

5.1. **The new embedding of $\mathcal{M}\mathrm{f}(J)$.** In this subsection we answer to the first question raised in the Introduction. In Theorem 5.4 we prove that the set of equations in $K[C]$ defining $\mathcal{M}\mathrm{f}(J)$ allows the elimination of a large number of parameters, more precisely those of $C \setminus \widetilde{C}$. Furthermore, using results of previous sections about the superminimal reduction, we are able to determine a set of equations defining $\mathcal{M}\mathrm{f}(J)$ in $K[\widetilde{C}]$ avoiding at all the introduction of parameters in $C \setminus \widetilde{C}$. This fact combined with the choice of a small set of S-polynomials (according to Corollary 4.6 or Theorem 4.7) will turn out to be significantly useful in projecting an effective algorithm for the computation of such equations. Furthermore, this new sets of equations turns out to be more suitable in order to compare marked schemes of m-truncation ideals of a strongly stable saturated ideal $\underline{J}$ as m varies.

Definition 5.2. Let $x^\alpha \in B_J$ and t be an integer such that $x_0^t \cdot x^\alpha \xrightarrow{s\mathcal{G}*} H_\alpha \in K[\tilde{C}, x]$, with H_α strongly reduced (the integer t exists by Theorem 3.14). We can write $H_\alpha = H'_\alpha + x_0^t \cdot H''_\alpha$, where no monomial appearing in H'_α is divisible by x_0^t. We will denote by:

- $\mathfrak{B} = \{C_{\alpha\gamma} - \phi_{\alpha\gamma} \ : \ x^\alpha \in B_J \setminus sB_J, x^\gamma \in \mathcal{N}(J)_{|\alpha|}\}$ the set of the x-coefficients of $T(F_\alpha) - H''_\alpha$ for every $x^\alpha \in B_J$;
- $\mathfrak{D}_1 \subset K[\widetilde{C}]$ the set of the x-coefficients of H'_α for every $x^\alpha \in B_J \setminus sB_J$;
- $\mathfrak{D}_2$ the set of the x-coefficients of the strongly reduced polynomials in $(s\mathcal{G})K[\widetilde{C}, x]$.

Remark 5.3. Observe that not only $\mathfrak{D}_2$ but also $\mathfrak{B}$ and $\mathfrak{D}_1$ are well-defined thanks to the uniqueness of H_α, by Theorem 3.14, (ii).

Theorem 5.4. *The J-marked scheme $\mathcal{M}\mathrm{f}(J)$ is defined by the ideal $\widetilde{\mathfrak{A}}_J := \mathfrak{A}_J \cap K[\widetilde{C}]$ as subscheme of the affine space $\mathbb{A}^{|\widetilde{C}|}$, where $|\widetilde{C}| = \sum_{x^\alpha \in sB_J} |\mathcal{N}(J)_{|\alpha|}|$. Moreover $\mathfrak{A}_J = (\mathfrak{B} \cup \mathfrak{D}_1 \cup \mathfrak{D}_2)K[C]$ and $\widetilde{\mathfrak{A}}_J = (\mathfrak{D}_1 \cup \mathfrak{D}_2)K[\widetilde{C}]$.*

Proof. For the first part it is sufficient to prove that $\mathfrak{A}_J$ contains $\mathfrak{B}$ and so it contains an element of the type $C_{\alpha\gamma} - \phi_{\alpha\gamma}$, for every $C_{\alpha\gamma} \in C \setminus \widetilde{C}$, where $\phi_{\alpha\gamma} \in K[\widetilde{C}]$, that allows the elimination of the variables $C_{\alpha\gamma} \in C \setminus \widetilde{C}$.

It is clear by the construction in Definition 5.2 that H_α belongs to $K[\widetilde{C}, x]$ and that both $x_0^t \cdot T(F_\alpha)$ and H_α are strongly reduced. Thus their difference $x_0^t \cdot T(F_\alpha) - H_\alpha$ is strongly reduced and moreover it belongs to $\mathfrak{I}_J$, because $x_0^t \cdot T(F_\alpha) - H_\alpha = -x_0^t \cdot F_\alpha + (x_0^t \cdot x^\alpha - H_\alpha)$. Hence, by Corollary 2.17, its x-coefficients belong to $\mathfrak{A}_J$ and in particular the coefficient of $x_0^t \cdot x^\gamma$ is of the type $C_{\alpha\gamma} - \phi_{\alpha\gamma}$, with $\phi_{\alpha\gamma} \in K[\widetilde{C}]$. Then $\mathfrak{A}_J \supseteq \mathfrak{B}$ and $\mathfrak{A}_J$ is generated by $\mathfrak{B} \cup \widetilde{\mathfrak{A}}_J$.

To prove the second part, it is sufficient to show that $\mathfrak{A}_J \cap K[\widetilde{C}] = (\mathfrak{D}_1 \cup \mathfrak{D}_2)K[\widetilde{C}]$.

"$\supseteq$" Taking the x-coefficients in $x_0^t \cdot T(F_\alpha) - H_\alpha$ of monomials that are not divisible by x_0^t, we see that $\mathfrak{A}_J$ contains the x-coefficients of H'_α. Then $\mathfrak{A}_J \cap K[\widetilde{C}] \supseteq \mathfrak{D}_1$, because $H'_\alpha \in K[\widetilde{C}, x]$.

Moreover we recall that $\mathfrak{A}_J$ is made by all the x-coefficients in the polynomials of $\mathfrak{I}_J$ that are strongly reduced. Indeed, $\mathfrak{A}_J$ is made by all the x-coefficients of the polynomials of $\mathfrak{I}_J$ that are J-reduced. Since the degree of the monomials in the variables x of every polynomial in $\mathfrak{I}_J$ is $\geq m$, then "J-reduced" is equivalent to "$\underline{J}$-reduced", that it is strongly reduced, by Lemma 3.2, (iv). Then $\mathfrak{A}_J \cap K[\widetilde{C}] \supseteq \mathfrak{D}_2$, because $(s\mathcal{G})K[\widetilde{C}, x] \subset \mathfrak{I}_J$.

"$\subseteq$" For every polynomial $F \in K[C,x]$, let us denote by F^ϕ the polynomial in $K[\widetilde{C},x]$ obtained substituting every $C_{\alpha\gamma} \in C \setminus \widetilde{C}$ by $\phi_{\alpha\gamma}$; if F is strongly reduced, then F^ϕ is strongly reduced too. Observe that for every $x^\alpha \in B_J$ we have $F^\phi_\alpha = x^\alpha - H''_\alpha$ and moreover $x_0^t(x^\alpha - H''_\alpha) - H'_\alpha \in (s\mathcal{G})K[\widetilde{C},x]$. In particular $x_0^t F^\phi_\alpha$ and $x_0^t(x^\alpha - H''_\alpha) - H'_\alpha$ are equal modulo $\mathfrak{D}_1$.

It remains to prove that every element $w \in \mathfrak{A}_J \cap K[\widetilde{C}]$ can be obtained modulo $\mathfrak{D}_1$ as a x-coefficient in some strongly reduced polynomial of the ideal $(s\mathcal{G}) \subset K[\widetilde{C}]$. We know that w is a x-coefficient in a strongly reduced polynomial $D \in \mathfrak{I}_J$.

If $D = \sum D_\alpha F_\alpha \in \mathfrak{I}_J$, then for a suitable t,

$$x_0^t \cdot D^\phi = \sum D^\phi_\alpha \cdot \left(x_0^t \cdot (x^\alpha - H''_\alpha) - H'_\alpha\right) \mod \mathfrak{D}_1$$

and the polynomial in the right-hand side of the equality is strongly reduced and it belongs to $(s\mathcal{G})K[\widetilde{C},x]$. Therefore w is still one of the x-coefficients of D^ϕ since it does not contain any variable in $C \setminus \widetilde{C}$ and it remains unchanged. Then $w \in (\mathfrak{D}_1 \cup \mathfrak{D}_2)K[\widetilde{C}]$. □

Proposition 5.5. *Let $\widetilde{\mathfrak{A}}_J$ be as in Theorem 5.4 and let $\mathfrak{U}$ be any ideal in $K[\widetilde{C}]$. Assume that $\mathfrak{U} \subseteq \widetilde{\mathfrak{A}}_J$ and that the following conditions hold:*

(i) For every monomial $x^\beta \in B_J \setminus sB_J$, $x^\beta = x^{\underline{\alpha}} *_{\underline{J}} x^\delta \in J$, there exists t such that we have a formula of type

$$x_0^t \cdot x^\beta = \sum b_i x^{\eta_i} F_{\alpha_i} + H_\beta,$$

with $a_i \in K[\widetilde{C}]$, $F_{\alpha_i} \in s\mathcal{G}$, $x^{\eta_i} \leq_{Lex} x^\delta$, $x^{\underline{\eta}_j + \underline{\alpha}_j} = x^{\underline{\alpha}_j} *_{\underline{J}} x^{\underline{\eta}_j}$ and $H_\beta = H'_\beta + x_0^t \cdot H''_\beta$, with H_β strongly reduced, x_0^t does not divide any monomial in $\mathrm{Supp}(H'_\beta)$ and $\mathrm{Coeff}_x(H'_\beta) \subseteq \mathfrak{U}$.

(ii) For every polynomial $F_\alpha \in s\mathcal{G}$ and for every $x_i > \min(x^{\underline{\alpha}})$ there exists t such that we have a formula of type

$$x_0^t x_i F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H_{i,\alpha}$$

where $b_j \in K[\widetilde{C}]$, $F_{\alpha_j} \in s\mathcal{G}$, $x^{\eta_j} <_{\mathtt{Lex}} x_i$, $x^{\underline{\eta}_j + \underline{\alpha}_j} = x^{\underline{\alpha}_j} *_{\underline{J}} x^{\underline{\eta}_j}$, $H_{i,\alpha}$ strongly reduced and $\mathrm{Coeff}_x(H_{i,\alpha}) \subseteq \mathfrak{U}$.

Then $\mathfrak{U} = (\mathfrak{D}_1 \cup \mathfrak{D}_2) = \widetilde{\mathfrak{A}}_J$.

Proof. Thanks to (i), we immediately have that $\mathfrak{D}_1 \subseteq \mathfrak{U}$.

For the inclusion $\mathfrak{D}_2 \subseteq \mathfrak{U}$, observe that if (i) and (ii) hold for $\mathfrak{U}$, then we can use the same arguments of Proposition 2.15 and obtain that:
for every $F_\alpha \in s\mathcal{G}$, for every x^δ, there exists t such that

$$x_0^t x^\delta F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H \tag{4}$$

with $b_j \in K[\widetilde{C}]$, $F_{\alpha_j} \in s\mathcal{G}$, $x^{\eta_j} <_{\mathtt{Lex}} x^\delta$, $x^{\underline{\eta}_j + \underline{\alpha}_j} = x^{\underline{\alpha}_j} *_{\underline{J}} x^{\underline{\eta}_j}$ $\mathrm{Supp}_x(H_{\delta,\alpha}) \subseteq \mathcal{N}(J)$ and $\mathrm{Coeff}_x(H_{\delta,\alpha}) \subseteq \mathfrak{U}$.

We can also prove the uniqueness of such a rewriting: thanks to the uniqueness of the decomposition by $*_{\underline{J}}$, the polynomials $x^{\eta_j} F_{\alpha_j}$ that can appear in (4) have pairwise different head terms. So an analogous of Corollary 2.16 holds for this setting.

Thanks to this uniqueness, as in Corollary 2.17, we get the non trivial inclusion of the thesis. □

Proposition 5.5 is very important from the computational point of view. Indeed, its condition (i) allows to explicitely construct the set of polynomials $\mathfrak{B}$, namely to write a J-marked set $\widetilde{\mathcal{G}}$ in $K[\widetilde{C},x]$, whose superminimal set is $s\mathcal{G}$. Using such a J-marked set in $K[\widetilde{C},x]$, we can use either Theorem 4.5 or Corollary 4.6 or Theorem 4.7 to obtain a set of generators for $\widetilde{\mathfrak{A}}_J$. For instance, the algorithm presented in the Appendix is based on Theorem 4.7 and the proof of its correctness on Proposition 5.5. In the future, we will investigate which is the best set of polynomials to start from in order to get a performing algorithm for the computation of equations for $\mathcal{M}\mathrm{f}(J)$. The correctness of such an algorithm will be verified by the conditions of Proposition 5.5.

5.2. **Relations among $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ as m varies.** In this subsection we will compare the marked schemes constructed from different truncations of a saturated strongly stable ideal $\underline{J}$. Let us consider two integers m', m, $(m' < m)$. If I is an ideal in the $\underline{J}_{\geq m'}$-marked family $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m'})$, then it is not difficult to show that $I_{\geq m}$ belongs to the marked family $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$, namely that there is a injective map of sets $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m'}) \to \mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$. Aim of the present subsection is a scheme theoretical version of this fact; indeed we will prove that there is a closed embedding of schemes $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m'}) \hookrightarrow \mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ that induces the previous one on the sets of closed points. It is sufficient to prove the existence of such a closed embedding for $m' = m-1$; in this case we denote the embedding map by ϕ_m. Furthermore, we characterize the cases in which ϕ_m is a isomorphism.

To this purpose, the main tool we will use is the set of defining equations for a J-marked scheme obtained by superminimal reduction, namely the ideal $\widetilde{\mathfrak{A}}_J$; moreover we will consider at the Zariski tangent space of $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ at the origin, denoted by $T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m}))$.

Remark 5.6. As for any affine variety, if $\mathcal{M}\mathrm{f}(J)$ is defined by an ideal $\mathfrak{U}$ as a subscheme of an affine space $\mathbb{A}^N$, then the Zariski tangent space $T_0(\mathcal{M}\mathrm{f}(J))$ is defined by the linear part of a set generators of $\mathfrak{U}$ so that it can be identified to a linear subspace of $\mathbb{A}^N$. In the special case of marked schemes, it is quite easy to compute a set of generators for $T_0(\mathcal{M}\mathrm{f}(J))$, using the properties and techniques of [15, Definition 3.4 and Proposition 4.3], [24, Proposition 3.4] and [10, Theorem 3.2].

Theorem 5.7 is inspired by an analogous result proved for Gröbner Strata in [15, Theorem 4.7]. Given a monomial ideal J, the *Gröbner Stratum* $\mathcal{S}t(J,\prec)$ of J w.r.t. a term order $\prec$ can be isomorphically projected in its Zariski tangent space at the origin $T_0(\mathcal{S}t(J,\prec))$ (see [15, Proposition 4.3]). Furthermore, if the origin is a smooth point, then $\mathcal{S}t(J,\prec)$ is isomorphic to this tangent space. Unluckily, it is not true that for every strongly stable ideal J there exists a term order $\prec$ such that $\mathcal{M}\mathrm{f}(J) \simeq \mathcal{S}t(J,\prec)$, as shown in [7, Appendix], so in general we cannot project isomorphically $\mathcal{M}\mathrm{f}(J)$ into $T_0(\mathcal{M}\mathrm{f}(J))$.

We introduce some useful notations: once fixed a saturated strongly stable ideal $\underline{J}$ and a positive integer m, we denote by

- $\mathcal{G}^{[m]}$ a $\underline{J}_{\geq m}$-marked set as in (1) and with $F^{[m]}_\beta$ a marked polynomial belonging to $\mathcal{G}^{[m]}$;
- $C^{[m]}$ the set of parameters $C^{[m]}_{\alpha\gamma}$ appearing in the tails of the marked polynomials in $\mathcal{G}^{[m]}$;
- $s\mathcal{G}^{[m]}$ the set of superminimals of $\mathcal{G}^{[m]}$;
- $\widetilde{C}^{[m]}$ the subset of $C^{[m]}$ containing only the parameters $\widetilde{C}^{[m]}_{\alpha\gamma}$ appearing in the tails of marked polynomials in $s\mathcal{G}^{[m]}$;
- $\widetilde{\mathfrak{A}}^{[m]}$ is the ideal in $K[\widetilde{C}^{[m]}]$ defining $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ as a subscheme in $\mathbb{A}^{|\widetilde{C}^{[m]}|}$ (defined in Theorem 5.4).

Theorem 5.7. *Let $\underline{J}$ be a saturated strongly stable ideal and let m be any integer. With the previous notations, the followings hold:*

(i) $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1})$ is a closed subscheme of $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ cut out by a suitable linear space. More precisely, $\widetilde{C}^{[m-1]}$ can be identified with a suitable subset of $\widetilde{C}^{[m]}$ so that the following diagram of schemes commutes:

$$\begin{tikzcd}
\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1}) \arrow[r, hook, "\phi_m"] \arrow[d, hook] & \mathcal{M}\mathrm{f}(\underline{J}_{\geq m}) \arrow[d, hook] \\
\mathbb{A}^{|\widetilde{C}^{[m-1]}|} \arrow[r, hook] & \mathbb{A}^{|\widetilde{C}^{[m]}|}
\end{tikzcd} \tag{5}$$

(ii) Let Ω be the number of monomials $x^\alpha \in B_{\underline{J}}$ of degree $m+1$ divisible by x_1 and $\Theta := |B_{\underline{J}} \cap S_{\leq m-1}|$; then,

$$\dim T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})) \geq \dim T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1})) + \Omega \cdot \Theta.$$

(iii) $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1}) \simeq \mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ if and only if either $\underline{J}_{\geq m-1} = \underline{J}_{\geq m}$ or no monomial of degree $m+1$ in $B_{\underline{J}}$ is divisible by x_1.

In particular:

$$\mathcal{M}\mathrm{f}(\underline{J}_{\geq \rho-1}) \simeq \mathcal{M}\mathrm{f}(\underline{J}_{\geq m}), \text{ for every } m \geq \rho$$

where ρ is the maximal degree of monomials divisible by x_1 in $B_{\underline{J}}$.

Proof.

(i) Thanks to Theorem 5.4, a $\underline{J}_{\geq m}$-marked scheme is defined by an ideal generated by polynomials of $K[\widetilde{C}^{[m]}]$ that are constructed using only the superminimals. So, now it is enough to prove that the set of superminimals $s\mathcal{G}^{[m-1]}$ corresponds to $s\mathcal{G}^{[m]}$ modulo a subset of the variables $\widetilde{C}^{[m]}$, in the following sense.

Consider $x^\alpha \in sB_{\underline{J}_{\geq m-1}}$. If $|\alpha| \geq m$, then x^α belongs to $sB_{\underline{J}_{\geq m}}$ and we can identify $F^{[m]}_\alpha \in s\mathcal{G}^{[m]}$ and $F^{[m-1]}_\alpha \in s\mathcal{G}^{[m-1]}$ (and in particular the variables in their tails: $\widetilde{C}^{[m]}_{\alpha\gamma} = \widetilde{C}^{[m-1]}_{\alpha\gamma}$).

If $|\alpha| = m-1$, then we can consider the corresponding superminimal element $F^{[m]}_\beta \in s\mathcal{G}^{[m]}$, with $x^\beta = x_0 \cdot x^\alpha$. Then we identify the variable $\widetilde{C}^{[m]}_{\beta\delta'}$, which is the coefficient of a monomial in $\mathrm{Supp}_x(F^{[m]}_\beta)$ of kind $x^{\delta'} = x_0 \cdot x^\delta$, with the variable $\widetilde{C}^{[m-1]}_{\alpha\delta}$ which is the coefficient of the monomial x^δ in $\mathrm{Supp}_x(F^{[m-1]}_\alpha)$.

We repeat this identifications for all $x^\alpha \in sB_{\underline{J}_{\geq m-1}}$ and we denote by $\overline{C}^{[m]}$ the subset of $\widetilde{C}^{[m]}$ containing the variables non-identified with variables of $\widetilde{C}^{[m-1]}$, that is the variables appearing as coefficients of monomials not divisible by x_0 in the tails of polynomials in $s\mathcal{G}^{[m]} \setminus s\mathcal{G}^{[m-1]}$. Now, every polynomial in $s\mathcal{G}^{[m]} \mod (\overline{C}^{[m]})$ either belongs to $s\mathcal{G}^{[m-1]}$ or is a polynomials of $s\mathcal{G}^{[m-1]}$ multiplied by x_0. Thanks to Theorem 5.4, we have that

$$\widetilde{\mathfrak{A}}^{[m]} + \left(\overline{C}^{[m]}\right) \simeq \widetilde{\mathfrak{A}}^{[m-1]}.$$

This relation among the ideals induces the embeddings of scheme of diagram (5).

(ii) We now consider $x^\gamma \in B_{\underline{J}}$, $|\gamma| = m+1$, x^γ divisible by x_1. We define $x^\beta := x^\gamma / x_1$; observe that $x^\beta \in \mathcal{N}(\underline{J})$. Furthermore, x^β is not divisible by x_0, otherwise x^γ would be too.

Then, for every $x^{\underline{\alpha}} \in B_{\underline{J}}$ with $|\underline{\alpha}| \leq m-1$, there is $F^{[m]}_{\alpha} = x^{\underline{\alpha}} x_0^{m-|\underline{\alpha}|} - T(F^{[m]}_\alpha) \in s\mathcal{G}^{[m]}$ such that $x^\beta \in \mathrm{Supp}_x(T(F^{[m]}_\alpha))$. We focus on the coefficient $\widetilde{C}^{[m]}_{\alpha\beta}$ of x^β. Since x^β is not divisible by x_0, $\widetilde{C}^{[m]}_{\alpha\beta}$ cannot be identified with a coefficient appearing in $F^{[m-1]}_\alpha = x^{\underline{\alpha}} x_0^{m-|\underline{\alpha}|-1} - T(F^{[m-1]}_\alpha) \in s\mathcal{G}^{[m-1]}$. So $\widetilde{C}^{[m]}_{\alpha\beta}$ belongs to the subset of variables $\overline{C}^{[m]}$ defined in the proof of (i).

We now use the construction of $T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m}))$ recalled in Remark 5.6. If we think about syzygies of the ideal $\underline{J}_{\geq m}$, we can see that in a S-polynomial, $F^{[m]}_\alpha$ is multiplied by a monomial x^δ

divisible by x_i, $i > 0$. In particular, $x^\delta \cdot x^\beta$ belongs to $\underline{J}_{\geq m}$: if $x_i = x_1$ we are done by construction, otherwise we apply the strongly stable property because $x_1 x^\beta x^\delta = \frac{x^\gamma}{x_i} x_1 x^\delta$ belongs to $\underline{J}_{\geq m}$. This means that the coefficient $\widetilde{C}^{[m]}_{\alpha\beta}$ does not appear in any equation defining $T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m}))$.

Applying this argument to the Ω monomials in $B_{\underline{J}}$ of degree $m+1$ which are divisible by x_1 and to the Θ monomials in $B_{\underline{J}}$ of degree $\leq m-1$, we obtain the result.

(iii) If $\underline{J}_{\geq m} = \underline{J}_{\geq m-1}$, obviously $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m}) = \mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1})$. We now assume that $\underline{J}_{\geq m} \neq \underline{J}_{\geq m-1}$ and no monomial of degree $m+1$ in the monomial basis of $\underline{J}$ is divisible by x_1; we prove that every polynomial in $s\mathcal{G}^{[m]}$ either belongs to $s\mathcal{G}^{[m-1]}$ or it is the product of x_0 by the "corresponding" polynomial in $s\mathcal{G}^{[m-1]}$.

If $x^\alpha \in sB_{\underline{J}_{\geq m-1}}$ and $|\alpha| \geq m$, then $F^{[m]}_\alpha \in s\mathcal{G}^{[m]}$ and $F^{[m-1]}_\alpha \in s\mathcal{G}^{[m-1]}$ have the same shape and we can identify them letting $\widetilde{C}^{[m]}_{\alpha\gamma} = \widetilde{C}^{[m-1]}_{\alpha\gamma}$, as done in the proof of (i). If $|\alpha| = m-1$, then $x^\beta = x_0 \cdot x^\alpha \in sB_{\underline{J}_{\geq m}}$ and all the monomials in the support of $x_0 \cdot F^{[m-1]}_\alpha$ appear in the support of $F^{[m]}_\beta$ (and we identify their coefficients as above). In the support of $F^{[m]}_\beta$ there are also some more monomials that are not divisible by x_0. We will prove now that the coefficients of these last monomials in fact belong to $\widetilde{\mathfrak{A}}^{[m]}$.

Consider the monomial $x_0 \cdot x_1 \cdot x^\alpha$. If we perform its reduction using $s\mathcal{G}^{[m]}$, the first step of reduction will lead to

$$x_0 \cdot x_1 \cdot x^\alpha \xrightarrow{s\mathcal{G}^{[m]}*} x_1 T(F^{[m]}_\beta).$$

Let x^γ be a monomial of $\mathrm{Supp}(T(F^{[m]}_\beta))$. If $x_1 \cdot x^\gamma \in \underline{J}_{\geq m}$, then $x_1 \cdot x^\gamma = x^{\underline{\alpha}'} *_{\underline{J}} x^\eta$, with $x^{\underline{\alpha}'} \in B_{\underline{J}}$ and $x^\eta <_{\mathtt{Lex}} x_1$. If $x^\eta = 1$, then $|\underline{\alpha}'| = m+1$ and $x^{\underline{\alpha}'}$ is divisible by x_1, against the hypothesis. Then $x^\eta = x_0^t$, with $t > 0$, and so the monomial $x_1 \cdot x^\gamma \in \underline{J}_{\geq m}$ is actually divisible by x_0. If $x_1 \cdot x^\gamma \in \mathcal{N}(\underline{J}_{\geq m})$, then this monomial is not further reducible, so that its coefficient belongs to $\widetilde{\mathfrak{A}}^{[m]}$.

Vice versa, by contradiction suppose now that $\underline{J}_{\geq m-1} \neq \underline{J}_{\geq m}$ and that there exists $x^\alpha \in B_{\underline{J}}$ divisible by x_1, $|\alpha| = m+1$. Using (ii), we have that $T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1})) \not\simeq T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m}))$ because $\dim T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1})) < \dim T_0(\mathcal{M}\mathrm{f}(\underline{J}_{\geq m}))$, and so $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m-1}) \not\simeq \mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$.

For the last part of the statement, note that if ρ is the maximal degree of a monomial divisible by x_1 in the monomial basis of $\underline{J}$, for every $m \geq \rho$, applying iteratively (iii) we obtain

$$\mathcal{M}\mathrm{f}(\underline{J}_{\geq \rho-1}) \simeq \mathcal{M}\mathrm{f}(\underline{J}_{\geq m}). \qquad \square$$

In the above setting, if $p(t)$ is the Hilbert polynomial of $S/\underline{J}$ and r is its Gotzmann number, it is worth considering the r-truncation of $\underline{J}$. Indeed, in [4] the authors prove that $\mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$ is naturally isomorphic to an open subset of the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$. We recall that r is the maximum among the regularities of ideals that are closed points of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$; hence, $r \geq \mathrm{reg}(\underline{J}) \geq \rho - 1$. Then Theorem 5.7 allows us to study such an open subset of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ embedded in an affine space of lower dimension than the expected one. More precisely:

Corollary 5.8. *Let $\underline{J}$ be a saturated strongly stable ideal, and let ρ be the maximal degree of monomials divisible by x_1 in $B_{\underline{J}}$. For every $m \geq \rho-1$, $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ can be embedded in an affine space of dimension*

$$|\widetilde{C}^{[\rho-1]}| = \sum_{x^\alpha \in sB_{\underline{J}_{\geq \rho-1}}} |\mathcal{N}(\underline{J})_{|\alpha|}| \leq |B_{\underline{J}}| \cdot p(r'), \tag{6}$$

where $r' = \mathrm{reg}(\underline{J})$ and $p(t)$ is the Hilbert polynomial of $S/\underline{J}$.

Proof. The equality of (6) directly follows from Theorem 5.7. For the inequality, we simply need to observe that the regularity r' of a strongly stable ideal is simply the maximum of the degrees of its monomial generators; hence every monomial in $sB_{\underline{J}_{\geq \rho-1}}$ has degree $\leq r'$. Furthermore r' is greater than or equal to the regularity of the Hilbert function of $S/\underline{J}$, thus $|\mathcal{N}(\underline{J})_{|\alpha|}| \leq \mathcal{N}(\underline{J}_{r'}) = p(r')$.

An equivalent proof follows from the diagram (5) of Theorem 5.7. $\square$

6. Examples

In the hypothesis that the field K has characteristic 0, the methods of computations developed in the previous sections can be applied to the study of Hilbert schemes: indeed, for m big enough, $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ corresponds to an open subset of the Hilbert scheme parameterizing the ideals having the same Hilbert polynomial as $S/\underline{J}$ (see [4]).

Now we give some examples for applications of the obtained results, mainly Theorem 4.5, Theorem 4.7 and Theorem 5.7. We keep on using the notations introduced before Theorem 5.7.

Example 6.1. Let $\underline{J}$ be the saturated strongly stable ideal $(x_n, \dots, x_2, x_1^{\mu}) \subseteq S = K[x_0, \dots, x_n]$. Observe that $\underline{J}$ is a `Lex`-segment, the Hilbert polynomial of $S/\underline{J}$ is $p(t) = \mu$, the regularity of $\underline{J}$ is $r' = \mu$, and also $\rho = \mu$. By Corollary 5.8, $\mathcal{M}\mathrm{f}(\underline{J})$ can be embedded into an affine space of dimension $2n-2+\mu$. Using the criterion of Theorem 4.5, we can see that actually $\mathcal{M}\mathrm{f}(\underline{J}) \simeq \mathbb{A}^{2n-2+\mu}$, as shown also in [24]. By Theorem 5.7, (iii), $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ is isomorphic to $\mathcal{M}\mathrm{f}(\underline{J})$ for every $m \leq \mu - 2$.

It is well-known that Proj $(S/\underline{J})$ is the `Lex`-point of $\mathcal{H}ilb^n_{\mu}$ and lies on a component of dimension $n\mu$ (see [21]). Then $\mathcal{M}\mathrm{f}(\underline{J}_{\geq m})$ is not isomorphic to an open subset of $\mathcal{H}ilb^n_{\mu}$ for every $m \leq \mu - 2$.

On the other hand, the same reasonings above leads to $\mathcal{M}\mathrm{f}(\underline{J}_{\geq \mu}) \simeq \mathbb{A}^{n\mu}$ so that $\mathcal{M}\mathrm{f}(\underline{J}_{\geq \mu})$ is an open subset of $\mathcal{H}ilb^n_{\mu}$. This is shown also in [24].

Example 6.2. We consider the strongly stable saturated ideal

$$\underline{J} = (x_2^3, x_1x_2^2, x_1^2x_2, x_1^5) \subseteq K[x_0, x_1, x_2].$$

It corresponds to a point of $\mathcal{H}ilb^2_8$, with Gotzmann number $r = 8$, the regularity r' of $\underline{J}$ is 5, and the same value for ρ. By Theorem 5.7 we have that $\mathcal{M}\mathrm{f}(\underline{J}_{\geq 4}) \simeq \mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$. Observe that $\underline{J}_{\geq 4}$ is not segment w.r.t. any term order (see [7, Appendix]), hence in this case the results of [15] do not apply.

In [7, Appendix], the authors first consider $\mathcal{M}\mathrm{f}(\underline{J}_{\geq 4})$ as an affine subscheme of A^{64} and then show that 45 of the variables can be eliminated, but using a time-consuming process of elimination of variables. By Corollary 5.8, we can directly embed $\mathcal{M}\mathrm{f}(\underline{J}_{\geq r})$ in an affine space of dimension 32, and we have to eliminate only 13 of the remaining variables.

Example 6.3. We take $p(t) = 4t$, $n = 3$, $q(t) = \binom{3+t}{3} - p(t) = \binom{3+t}{3} - 4t$; the Gotzmann number of $p(t)$ is $r = 6$. The Hilbert scheme $\mathcal{H}\mathrm{ilb}^3_{4t}$ can be considered as a subscheme of the Grassmannian $\mathbb{G} = \mathbb{G}(q(6), K[x]_6)$ of linear spaces of dimension $q(6) = 60$ in the vector space $K[x]_6$ of dimension $\binom{3+6}{3} = 84$ (see [4, Section 1] for some details about this construction). Therefore equations for $\mathcal{H}\mathrm{ilb}^3_{4t}$ involve $E = \binom{84}{60} - 1 \sim 6 \cdot 10^{20}$ Plücker coordinates. We can obtain an open cover of $\mathcal{H}\mathrm{ilb}^3_{4t}$ by the non-vanishing of each Plücker coordinate of $\mathbb{G}$: we get E open subsets, each of them isomorphic to a subscheme of $\mathbb{A}^{1440}$.

In [4] the authors consider a different open cover (up to the action of $PGL(4)$) of $\mathcal{H}\mathrm{ilb}^3_{4t}$, formed by 4 open subsets only, isomorphic to a marked scheme of a suitable truncation of the saturated strongly stable monomial ideals $\underline{J}_i$, $i = 1, 2, 3, 4$ in $K[x]$. We can choose for every i the truncation $m = r = 6$, nevertheless in order to perform computations with a lower number of variables, it is better to choose the truncations according to Theorem 5.7.

We denote the cardinality of the monomial basis of $\underline{J}_i$ by σ_i. In the following table we list the dimensions of the different affine spaces where we can embed the marked schemes, using Theorem 5.7

and Corollary 5.8.

	Monomial basis of $\underline{J}_i$	$\text{reg}(\underline{J}_i)$	σ_i	$\rho_i - 1$	$\sigma_i p\big(\text{reg}(\underline{J}_i)\big)$	$\widetilde{C}^{[\rho_i-1]}$
$\underline{J}_1$	x_3^2, x_3x_2, x_2^3	3	3	-1	36	28
$\underline{J}_2$	$x_3^2, x_3x_2, x_3x_1^2, x_2^4$	4	4	2	64	44
$\underline{J}_3$	$x_3^2, x_3x_2, x_3x_1, x_2^5, x_2^4x_1$	5	5	4	100	88
$\underline{J}_4$	$x_3, x_2^5, x_2^4x_1^2$	6	3	5	72	64

Observe that for $\underline{J}_1$ and $\underline{J}_2$, the truncation giving an open subset of $\mathcal{H}\text{ilb}^3_{4t}$ is exactly the saturated ideal.

$\underline{J}_4$ is the `Lex`-segment ideal: $\mathcal{M}\text{f}((\underline{J}_4)_{\geq 5})$ is isomorphic to $\mathbb{A}^{23}$ (see [15, Theorem 7.3]). In this case we should further eliminate 41 variables. This means that our bounds are not in general "sharp", however the computational consequences of Theorem 5.7 are significant and our results allow the treatment of non-trivial cases that cannot be handled with "classical" techniques.

Appendix. A pseudocode description of the algorithm for computing a J-marked scheme

We now describe a prototype of the algorithm for computing J-marked families based on Proposition 5.5 and on the analogous of the set L_1 defined in Theorem 4.7. The ideal J is always supposed to be a strongly stable m-truncation ideal.

Let us suppose that the following functions are made available.

- generators(J). It determines the monomial basis of J.
- superminimalGenerators(J). It determines the superminimal generators of J.
- superminimalReduction(H, sG). Given a J-marked superminimal set sG and a polynomial H, it returns a pair (t, h) where t is the minimal power of x_0 such that there is a superminimal reduction of $x_0^t H$ to a strongly reduced polynomial and h is such polynomial, namely $x_0^t H \xrightarrow{sG*} h$ (as in Theorem 3.14, (ii)).
- quotientAndRemainder(H, t). Given a polynomial H and a non-negative integer t, it returns the pair of polynomials (H', H'') such that $H = H' + H'' x_0^t$.
- pairsL1(sG). Given a J-marked superminimal set sG, it computes the pairs of polynomials belonging to the set $\mathcal{L}_1$ (analogous of the set L_1 of Theorem 4.7).
- coeff(H, x^α). It returns the coefficient of the monomial x^α in the polynomial H (obviously 0 if $x^\alpha \notin \text{Supp}(H)$).

1: MARKEDSCHEME(J)
Input: $J \subset K[x_0, \dots, x_n]$ strongly stable m-truncation ideal.
Output: an ideal defining the marked scheme $\mathcal{M}\mathrm{f}(J)$.
2: $B_J \leftarrow$ GENERATORS(J);
3: $sB_J \leftarrow$ SUPERMINIMALGENERATORS(J);
4: $\widetilde{\mathcal{G}} \leftarrow \emptyset$; $s\mathcal{G} \leftarrow \emptyset$;
5: **for all** $x^\alpha \in sB_J$ **do**
6: $\quad F_\alpha \leftarrow x^\alpha$;
7: $\quad$ **for all** $x^\beta \in \mathcal{N}(J)_{|\alpha|}$ **do**
8: $\quad\quad F_\alpha \leftarrow F_\alpha + \widetilde{C}_{\alpha\beta} x^\beta$;
9: $\quad$ **end for**
10: $\quad \widetilde{\mathcal{G}} \leftarrow \widetilde{\mathcal{G}} \cup \{F_\alpha\}$;
11: $\quad s\mathcal{G} \leftarrow s\mathcal{G} \cup \{F_\alpha\}$;
12: **end for**
13: equations $\leftarrow \emptyset$;
14: $\overline{B}_J \leftarrow B_J \setminus sB_J$;
15: **for all** $x^\alpha \in \overline{B}_J$ **do**
16: $\quad (t, H) \leftarrow$ SUPERMINIMALREDUCTION($x^\alpha, s\mathcal{G}$);
17: $\quad (H', H'') \leftarrow$ QUOTIENTANDREMAINDER(H, t);
18: $\quad$ **for all** $x^\eta \in \mathrm{Supp}(H')$ **do**
19: $\quad\quad$ equations $\leftarrow$ equations $\cup \{$COEFF(H', x^η)$\}$;
20: $\quad$ **end for**
21: $\quad \widetilde{\mathcal{G}} \leftarrow \widetilde{\mathcal{G}} \cup \{x^\alpha - H''\}$;
22: **end for**
23: $\mathcal{L}_1 \leftarrow$ PAIRSL1($s\mathcal{G}$);
24: **for all** $(F_\alpha, F_{\alpha'}) \in \mathcal{L}_1$ **do**
25: $\quad (t, H) \leftarrow$ SUPERMINIMALREDUCTION$\big(S(F_\alpha, F_{\alpha'}), s\mathcal{G}\big)$;
26: $\quad$ **for all** $x^\eta \in \mathrm{Supp}(H)$ **do**
27: $\quad\quad$ equations $\leftarrow$ equations $\cup \{$COEFF(H, x^η)$\}$;
28: $\quad$ **end for**
29: **end for**
30: **return** (equations);

Theorem A.1. *The algorithm* MARKEDSCHEME *is correct.*

Proof. To prove that the algorithm terminates it is sufficient to recall that the superminimal reduction is Noetherian (Theorem 3.14 (i)).

Now we show that the algorithm MARKEDSCHEME returns a set of generators for the ideal defining $\mathcal{M}\mathrm{f}(J)$. The starting point is the J-marked superminimal set $s\mathcal{G}$ given in Definition 5.1 , having parameters in $\widetilde{C}$ as coefficients of every monomial in the tails and get a set equations of polynomials in $K[\widetilde{C}]$. We claim that the ideal $\mathfrak{U}$ generated by equations coincides with the ideal $\widetilde{\mathfrak{A}}_J$ of Theorem 5.4, by Proposition 5.5.

Indeed in the first part (lines 15-22), the algorithm computes the superminimal reduction H of each monomial $x^\alpha \in B_J \setminus sB_J$ and it imposes the conditions required by Proposition 5.5 (i), in other words the algorithm computes the set $\mathfrak{D}_1 \subseteq \widetilde{\mathfrak{A}}_J$ of Definition 5.2. At the same time, the algorithm constructs the J-marked set $\widetilde{\mathcal{G}} \subset K[\widetilde{C}, x]$.

In the second part (lines 23-29), the algorithm considers pairs of superminimal generators $(F_\alpha, F_{\alpha'})$ such that $x_i x^\alpha = x^{\underline{\alpha'}} *_{\underline{J}} x^\eta$. Recall that $x^\eta <_{\texttt{Lex}} x_i$ by Lemma 2.5. These couples of polynomials in $s\mathcal{G}$ correspond to the couples of the set L_1 in Theorem 4.7.

At line 25 of the algorithm we compute the superminimal reduction of the associated S-polynomial

$$x_0^t S(F_\alpha, F'_\alpha) = x_0^t \left(x_i x_0^{t'} F_\alpha - x^\eta x_0^{t''} F_{\alpha'} \right) \xrightarrow{s\mathcal{G}*} H \qquad x_i x_0^{t'} = \frac{\mathrm{lcm}(x^\alpha, x^{\alpha'})}{x^\alpha},\ x^\eta x_0^{t''} = \frac{\mathrm{lcm}(x^\alpha, x^{\alpha'})}{x^{\alpha'}}$$

that is applying Lemma 4.3

$$x_0^t (x_0^{t'} x_i F_\alpha - x_0^{t''} x^\eta F_{\alpha'}) - \sum b_j x^{\eta_j} F_{\beta_j} = H$$

with $b_j \in K[\widetilde{C}]$, $F_{\beta_j} \in s\mathcal{G}$, $x^{\eta_j} <_{\texttt{Lex}} x_i x_0^{t'}$ and $x^{\underline{\eta_j}} <_{\texttt{Lex}} x_i$, so that

$$x_0^{\bar{t}} x_i F_\alpha = x^{\overline{\eta}} F_{\alpha'} + \sum b_j x^{\eta_j} F_{\beta_j} + H.$$

The polynomial H is strongly reduced and it belongs to the ideal $(s\mathcal{G}) \subseteq K[\widetilde{C}, x]$, then its x-coefficients belong to $\mathfrak{D}_2 \subseteq \widetilde{\mathfrak{A}}_J$.

Then by construction (lines 26-28), $\mathfrak{U}$ is contained in $\widetilde{\mathfrak{A}}_J$ and it satisfies the condition required by Proposition 5.5 (ii), hence $\mathfrak{U} = \widetilde{\mathfrak{A}}_J$. □

We are convinced that this version can be strongly strengthened drawing inspiration from some of the improvements studied for the computation of Gröbner bases and border bases. In this direction, we have already developed a first prototype which is giving good and promising results. In the following table, we report the results of the computation of the marked schemes considered in Example 6.3. The algorithm has been run on a MacBook Pro with a 2,4 GHz Intel Core 2 Duo processor.

	Parameters	Equations	Time
$\mathcal{M}\mathrm{f}(\underline{J}_1)$	28	28	0.165568 seconds
$\mathcal{M}\mathrm{f}(\underline{J}_2)$	44	64	0.295802 seconds
$\mathcal{M}\mathrm{f}\big((\underline{J}_3)_{\geq 4}\big)$	88	228	110050 seconds

The prototype of the algorithm is available at
www.personalweb.unito.it/paolo.lella/ HSC/Documents/MarkedSchemes.m2

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